

BROWNIAN EARTHWORM

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ABSTRACT. We prove that the distance between two reflected Brownian motions outside a sphere in a 3-dimensional flat torus does not converge to 0, a.s., if the radius of the sphere is sufficiently small, relative to the size of the torus.

1. INTRODUCTION

This article is partly motivated by a natural phenomenon. We would like to analyze the effect of a randomly moving earthworm on the soil. The soil is pushed aside by the earthworm. What is the asymptotic distribution of soil particles when time goes to infinity? Is the soil compacted or are soil particles more or less evenly spread over the region, especially when the earthworm is small compared to the size of the region? In our toy model, the earthworm is represented by a sphere following a Brownian path. We will next state the model in rigorous terms and then present a theorem and some conjectures. We will also briefly review related results.

Let D_1 be the flat d -dimensional torus with side length 1, i.e., D_1 is the cube $\{(x_1, \dots, x_d) \in \mathbb{R}^d : |x_k| \leq 1 \text{ for } k = 1, \dots, d\}$, with the opposite sides identified in the usual way. Let $\mathcal{B}(x, r)$ denote the open ball with center x and radius r . For $0 < r < 1$, let $D = D_1 \setminus \overline{\mathcal{B}(0, r)}$. Let $\mathbf{n}(x)$ denote the unit inward normal vector at $x \in \partial D = \partial \mathcal{B}(0, r)$. Let B be a standard d -dimensional Brownian motion, $x_0, y_0 \in \overline{D}$, $x_0 \neq y_0$, and consider the following Skorokhod equations,

$$X_t = x_0 + B_t + \int_0^t \mathbf{n}(X_s) dL_s^X \quad \text{for } t \geq 0, \quad (1.1)$$

$$Y_t = y_0 + B_t + \int_0^t \mathbf{n}(Y_s) dL_s^Y \quad \text{for } t \geq 0. \quad (1.2)$$

Here L^X is the local time of X on ∂D . In other words, L^X is a non-decreasing continuous process which does not increase when X is in D , i.e., $\int_0^\infty \mathbf{1}_D(X_t) dL_t^X = 0$, a.s. Equation (1.1) has a unique pathwise solution (X, L^X) such that $X_t \in \overline{D}$ for all $t \geq 0$ (see [9]). The reflected Brownian motion X is a strong Markov process. The same remarks apply to (1.2),

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so (X, Y) is also strong Markov. Note that on any time interval (s, t) such that $X_u \in D$ and $Y_u \in D$ for all $u \in (s, t)$, we have $X_u - Y_u = X_s - Y_s$ for all $u \in (s, t)$.

For $x, y \in D_1$, we use $\text{dist}(x, y)$ to denote the geodesic distance between x and y in the torus D_1 .

Theorem 1.1. *If the dimension $d = 3$ then there is $r_0 > 0$ such that for every $r \leq r_0$ and every $x_0 \neq y_0$, we have $\limsup_{t \rightarrow \infty} \text{dist}(X_t, Y_t) > 0$, a.s.*

An analogous problem was considered in [4] for domains $D \subset \mathbb{R}^2$. It was proved that if D is a bounded domain with a smooth boundary and at most one hole, then $\lim_{t \rightarrow \infty} \text{dist}(X_t, Y_t) = 0$, a.s. It is not known whether for any 2-dimensional domain D , $\limsup_{t \rightarrow \infty} \text{dist}(X_t, Y_t) > 0$ with positive probability.

Note that by the pathwise uniqueness of the solutions to (1.1)-(1.2), 0 is an absorbing state for the distance process $\text{dist}(X_t, Y_t)$; that is, if $\text{dist}(X_{t_0}, Y_{t_0}) = 0$, then $\text{dist}(X_t, Y_t) = 0$ for all $t \geq t_0$. Theorem 1.1 says that $\text{dist}(X_t, Y_t)$ never enters the absorbing state 0 nor converges to 0 as $t \rightarrow \infty$. Since D is compact, this suggests that $\text{dist}(X_t, Y_t)$ fluctuates and is a “recurrent” process. We suspect that (X_t, Y_t) has a stationary probability distribution but this does not follow from recurrence alone. Hence, we propose the following

Conjecture 1.2. *If the dimension $d = 3$ then there is $r_0 > 0$ such that for $r \leq r_0$ the process (X, Y) has a stationary distribution Q which does not charge the diagonal $\{(x, x) : x \in \overline{D}\}$. There is only one stationary distribution for (X, Y) which does not charge the diagonal.*

Since (1.1)-(1.2) have a unique pathwise solution, if $x_0 = y_0$ then $X_t = Y_t$ for all $t \geq 0$, a.s. It follows that (X, Y) has a unique stationary distribution Q' supported on the diagonal, characterized by the fact that the distribution of X under Q' is uniform in D .

Our state space D for reflected Brownian motion is a subset of a torus because three dimensional Brownian motion is transient so the result analogous to Theorem 1.1 for the complement of a ball in \mathbb{R}^3 is not interesting.

Problem 1.3. *Is Theorem 1.1 valid when the dimension $d = 2$?*

The reader may find it paradoxical that we can prove Theorem 1.1 in 3 dimensions but the analogous result in 2 dimensions is stated as an open problem. The reason is that the proof depends in a crucial way on the sign of a certain “Lyapunov exponent” $\lambda_\rho^* = 1 + \lambda_\rho$ where $\rho := 1/r$ and λ_ρ is defined in Theorem 4.1(ii) relative to the domain D . We prove in Lemma 4.2 that λ_ρ^* is positive for D if $d = 3$ and ρ is large. In the 2-dimensional case, the analogous exponent is equal to 0 ([4, Prop. 2.3]) and this critical value makes the problem hard. We could have defined the domain D as $D_1 \setminus A$, with A being not necessarily a

ball. It is easy to see that for many sets A , for example, those that are bounded, smooth and close to a polyhedron, λ^* is negative. It was shown in [4] that in 2-dimensional space, negative λ^* implies that $\lim_{t \rightarrow \infty} \text{dist}(X_t, Y_t) = 0$, a.s. In such a case, (X, Y) does not have a stationary distribution with some mass outside the diagonal. It is not known whether there is a 2-dimensional domain, bounded or unbounded, with positive λ^* . It is also not known whether for any 2-dimensional domain D , $\limsup_{t \rightarrow \infty} \text{dist}(X_t, Y_t) > 0$ with positive probability. Theorem 1.1 shows that this is the case for a subset of a three-dimensional torus. We believe that the theorem also holds in some bounded subsets of \mathbb{R}^3 but we will not provide a rigorous proof. We make this claim more precise in the following conjecture.

Conjecture 1.4. *Suppose that $\mathcal{B}(x_j, r) \subset \mathcal{B}(0, 1)$ for $j = 1, \dots, k$, and let $D_2 = \mathcal{B}(0, 1) \setminus \bigcup_{j=1}^k \overline{\mathcal{B}(x_j, r)} \subset \mathbb{R}^3$. If k is sufficiently large and $(\min_{1 \leq j \leq k} (1 - |x_j|) + \min_{1 \leq i < j \leq k} |x_i - x_j|) / r$ is sufficiently large, then Theorem 1.1 holds for D_2 .*

Suppose that Conjecture 1.2 is true, i.e., for some $r_0 > 0$ and all $r \leq r_0$, the process (X, Y) has a stationary distribution Q which does not charge the diagonal. This stationary measure Q depends on r , the radius of the ball deleted from the torus D_1 , so we can write Q_r to emphasize this dependence.

Conjecture 1.5. *Measures Q_r converge to the uniform probability distribution on $(D_1)^2$ when $r \rightarrow 0$.*

Next, we consider the flow X_t^x of reflected Brownian motions, defined for $x \in \overline{D}$ by

$$X_t^x = x + B_t + \int_0^t \mathbf{n}(X_s^x) dL_s^x, \quad \text{for } t \geq 0. \quad (1.3)$$

Here L^x is the local time of X^x on ∂D . Equations (1.3) have unique pathwise solution (X^x, L^x) for all x simultaneously because the construction of the solution to the Skorokhod equation given in [9] is deterministic. Let $|A|$ denote the Lebesgue measure of a set A and $\mathbf{Q}_{r,t}(A) = |\{x \in D : X_t^x \in A\}|$.

Conjecture 1.6. *Measures $\mathbf{Q}_{r,t}$ converge to a random measure \mathbf{Q}_r on $D_1 \setminus \mathcal{B}(0, r)$ when $t \rightarrow \infty$, in the sense of weak convergence of random measures. Random measures \mathbf{Q}_r converge weakly to the uniform measure on D_1 when $r \rightarrow 0$, in probability.*

In the context of (1.3), the earthworm picture is obtained by interpreting $\mathcal{B}(0, r) - B_t$ as a Brownian earthworm and $X_t^x - B_t$ as the location of a displaced soil particle.

For an extensive review of related results, see [3]. Some of those results will be recalled in Section 2.4. The present article is, philosophically speaking, a mirror image of [4]. That article analyzed domains where $\text{dist}(X_t, Y_t)$ converged to 0, while the present article analyzes

domains where the opposite is true. It was proved in [7, 8] that under mild technical assumptions on the domain, reflected Brownian motions X and Y do not coalesce in a finite time. A series of papers by Pilipenko [11, 12, 13] discuss stochastic flows of reflected processes. The article [14] is posted on Math ArXiv; it is a review and discussion of Pilipenko's previously published results.

The rest of the paper is organized as follows. Section 2 is a review of known results needed in this paper, including a review of excursion theory in Section 2.3 and some technical estimates from [5, 3] in Section 2.4. The proof of Theorem 1.1 is given in Section 3; it consists of several lemmas. The paper is based in an essential way on the exact and explicit evaluation of an integral representing the Lyapunov exponent λ_ρ . The calculation is rather tedious so it is relegated to Section 4.

2. PRELIMINARIES

2.1. General. For a process Z , set A and point a , let $T_A^Z = \inf\{t \geq 0 : Z_t \in A\}$, $T_a^Z = \inf\{t \geq 0 : Z_t = a\}$ and $\tau_A^Z = \inf\{t \geq 0 : Z_t \notin A\}$. By the Brownian scaling, if $\{X_t; t \geq 0\}$ is the reflecting Brownian motion on $D_1 \setminus \overline{\mathcal{B}(0, r)}$ driven by Brownian motion B_t , then $\{r^{-1}X_{r^2t}; t \geq 0\}$ is the reflecting Brownian motion on $(r^{-1}D_1) \setminus \overline{\mathcal{B}(0, 1)}$ driven by Brownian motion $r^{-1}B_{r^2t}$. For notational convenience, throughout the remaining part of this paper, we fix $\rho = 1/r > 1$ and take D_1 to be the flat d -dimensional torus with side length $2\rho > 2$, i.e., D_1 is the cube $\{(x_1, \dots, x_d) \in \mathbb{R}^d : |x_k| \leq \rho, k = 1, \dots, d\}$, with the opposite sides identified in the usual way, and let $D = D_1 \setminus \overline{\mathcal{B}(0, 1)}$.

2.2. Linear structure in torus. In Section 1, we used notation normally reserved for elements of linear spaces, such as vector sum (e.g., $X_s - Y_s$) and norm (e.g., $|X_t - Y_t|$). We will now make this convention precise. Note that the torus D_1 can be represented as the quotient $(\mathbb{R}/(2\rho\mathbb{Z}))^3$. For $x \in D_1$, let A_x denote the set of all points in \mathbb{R}^3 which correspond to x . For $x, y \in D_1$, we choose $x_1 \in A_x$ and $y_1 \in A_y$ with the minimal distance $|x_1 - y_1|$ among all such pairs. Then we let $x - y = x_1 - y_1$ and $|x - y| = |x_1 - y_1|$.

2.3. Review of excursion theory. This section contains a brief review of excursion theory needed in this paper. See, e.g., [10] for the foundations of the theory in the abstract setting and [2] for the special case of excursions of Brownian motion. Although [2] does not discuss reflected Brownian motion, all results we need from that book readily apply in the present context. We will use two different but closely related “exit systems.” The first one, presented below, is a simple exit system representing excursions of a single reflected Brownian motion from ∂D . The second exit system, introduced in Section 4, is more complex as it encodes the information about two processes. Our review applies to general domains D with smooth

boundaries but we will assume that D is the torus with a removed unit ball, as in Theorem 1.1.

Let \mathbb{P}^{x_0} denote the distribution of the process X defined by (1.1) and let \mathbb{E}^{x_0} be the corresponding expectation. Let \mathbb{P}_D^x denote the distribution of Brownian motion starting from $x \in D$ and killed upon exiting D .

An “exit system” for excursions of the reflected Brownian motion X from ∂D is a pair (L_t^*, H^x) consisting of a positive continuous additive functional L_t^* and a family of “excursion laws” $\{H^x\}_{x \in \partial D}$. We will soon show that $L_t^* = L_t^X$. Let Δ denote the “cemetery” point outside \overline{D} and let \mathcal{C} be the space of all functions $f : [0, \infty) \rightarrow \overline{D} \cup \{\Delta\}$ which are continuous and take values in \overline{D} on some interval $[0, \zeta)$, and are equal to Δ on $[\zeta, \infty)$. For $x \in \partial D$, the excursion law H^x is a σ -finite (positive) measure on \mathcal{C} , such that the canonical process is strong Markov on (t_0, ∞) , for every $t_0 > 0$, with the transition probabilities \mathbb{P}_D^x . Moreover, H^x gives zero mass to paths which do not start from x . We will be concerned only with the “standard” excursion laws; see Definition 3.2 of [2]. For every $x \in \partial D$ there exists a unique standard excursion law H^x in D , up to a multiplicative constant.

Excursions of X from ∂D will be denoted e or e_s , i.e., if $s < u$, $X_s, X_u \in \partial D$, and $X_t \notin \partial D$ for $t \in (s, u)$ then $e_s = \{e_s(t) = X_{t+s}, t \in [0, u-s)\}$ and $\zeta(e_s) = u-s$. By convention, $e_s(t) = \Delta$ for $t \geq \zeta$, so $e_t \equiv \Delta$ if $\inf\{s > t : X_s \in \partial D\} = t$. Let $\mathcal{E}_u = \{e_s : s \leq u\}$.

Let $\sigma_t = \inf\{s \geq 0 : L_s^* \geq t\}$ and let I be the set of left endpoints of all connected components of $(0, \infty) \setminus \{t \geq 0 : X_t \in \partial D\}$. The following is a special case of the exit system formula of [10]. For every $x \in \overline{D}$,

$$\mathbb{E}^x \left(\sum_{t \in I} V_t \cdot f(e_t) \right) = \mathbb{E}^x \int_0^\infty V_{\sigma_s} H^{X(\sigma_s)}(f) ds = \mathbb{E}^x \int_0^\infty V_t H^{X_t}(f) dL_t^*, \quad (2.1)$$

where V_t is a predictable process and $f : \mathcal{C} \rightarrow [0, \infty)$ is a universally measurable function which vanishes on excursions e_t identically equal to Δ . Here and elsewhere $H^x(f) = \int_{\mathcal{C}} f dH^x$.

The normalization of the exit system is somewhat arbitrary, for example, if (L_t^*, H^x) is an exit system and $c \in (0, \infty)$ is a constant then $(cL_t^*, (1/c)H^x)$ is also an exit system. One can even make c dependent on $x \in \partial D$. Theorem 7.2 of [2] shows how to choose a “canonical” exit system; that theorem is stated for the usual planar Brownian motion but it is easy to check that both the statement and the proof apply to the reflected Brownian motion. According to that result, we can take L_t^* to be the continuous additive functional whose Revuz measure is a constant multiple of the surface area measure on ∂D and H^x ’s to be standard excursion laws normalized so that

$$H^x(A) = \lim_{\delta \downarrow 0} \frac{1}{\delta} \mathbb{P}_D^{x+\delta \mathbf{n}(x)}(A), \quad (2.2)$$

for any event A in a σ -field generated by the process on an interval $[t_0, \infty)$, for any $t_0 > 0$. The Revuz measure of L^X is the measure $dx/(2|D|)$ on ∂D , i.e., if the initial distribution of X is the uniform probability measure μ in D then $\mathbb{E}^\mu \int_0^1 \mathbf{1}_A(X_s) dL_s^X = \int_A dx/(2|D|)$ for any Borel set $A \subset \partial D$. It has been shown in [4] that $L_t^* = L_t^X$.

2.4. Differentiability of stochastic flow of reflected Brownian motions. It was proved in [1, 3, 14], in somewhat different settings, that the stochastic flow of reflected Brownian motions is differentiable in the initial condition. We will use this result and we will also need a key estimate from [3] that was partly developed in [5]. First, we will recall some notation from [3]. The notation may seem somewhat awkward in the present context because it was developed for complicated arguments. We leave most of this notation unchanged to help the reader consult the results in [3].

We consider ∂D to be a smooth, properly embedded, orientable hypersurface (i.e., submanifold of codimension 1) in \mathbb{R}^3 , endowed with a smooth unit normal inward vector field \mathbf{n} . We consider ∂D as a Riemannian manifold with the induced metric. We use the notation $\langle \cdot, \cdot \rangle$ for both the Euclidean inner product on \mathbb{R}^n and its restriction to the tangent space $\mathcal{T}_x \partial D$ for any $x \in \partial D$, and $|\cdot|$ for the associated norm. For any $x \in \partial D$, let $\pi_x: \mathbb{R}^n \rightarrow \mathcal{T}_x \partial D$ denote the orthogonal projection onto the tangent space $\mathcal{T}_x \partial D$, so $\pi_x \mathbf{z} = \mathbf{z} - \langle \mathbf{z}, \mathbf{n}(x) \rangle \mathbf{n}(x)$, and let $\mathcal{S}(x): \mathcal{T}_x \partial D \rightarrow \mathcal{T}_x \partial D$ denote the shape operator (also known as the Weingarten map), which is the symmetric linear endomorphism of $\mathcal{T}_x \partial D$ associated with the second fundamental form. It is characterized by $\mathcal{S}(x)\mathbf{v} = -\partial_{\mathbf{v}} \mathbf{n}(x)$ for $\mathbf{v} \in \mathcal{T}_x \partial D$, where $\partial_{\mathbf{v}}$ denotes the ordinary Euclidean directional derivative in the direction of \mathbf{v} .

Recall that Δ is an extra ‘‘cemetery point’’ outside \overline{D} , so that we can send processes killed at a finite time to Δ . For $s \geq 0$ such that $X_s \in \partial D$ we let $\zeta(e_s) = \inf\{t > 0 : X_{s+t} \in \partial D\}$. Here e_s is an excursion starting at time s , i.e., $e_s = \{e_s(t) = X_{t+s}, t \in [0, \zeta(e_s))\}$. We let $e_s(t) = \Delta$ for $t \geq \zeta(e_s)$, so $e_t \equiv \Delta$ if $\zeta(e_s) = 0$.

Let σ_t^X be the inverse of local time L_t^X , i.e., $\sigma_t^X = \inf\{s \geq 0 : L_s^X \geq t\}$, and $\mathcal{E}_b = \{e_s : s < \sigma_b^X\}$. For $b, \varepsilon > 0$, let $\{e_{u_1}, e_{u_2}, \dots, e_{u_m}\}$ be the set of all excursions $e \in \mathcal{E}_b$ with $|e(0) - e((\zeta \wedge \sigma_b^X)-)| \geq \varepsilon$. We assume that excursions are labeled so that $u_k < u_{k+1}$ for all k and we let $\ell_k = L_{u_k}^X$ for $k = 1, \dots, m$. We also let $u_0 = \inf\{t \geq 0 : X_t \in \partial D\}$, $\ell_0 = 0$, $\ell_{m+1} = r$, and $\Delta \ell_k = \ell_{k+1} - \ell_k$. Let $x_k = e_{u_k}(\zeta \wedge \sigma_b^X)-$ be the right endpoint of excursion e_{u_k} for $k = 1, \dots, m$, and $x_0 = X_{u_0}$.

For $\mathbf{v}_0 \in \mathbb{R}^n$, let

$$\mathbf{v}_b = \exp(\Delta \ell_m \mathcal{S}(x_m)) \pi_{x_m} \cdots \exp(\Delta \ell_1 \mathcal{S}(x_1)) \pi_{x_1} \exp(\Delta \ell_0 \mathcal{S}(x_0)) \pi_{x_0} \mathbf{v}_0. \quad (2.3)$$

Note that all concepts based on excursions e_{u_k} depend implicitly on $\varepsilon > 0$, which is often suppressed in the notation. Let $\mathcal{A}_b^\varepsilon$ denote the linear mapping $\mathbf{v}_0 \rightarrow \mathbf{v}_b$.

It was proved in Theorem 3.2 in [5] that for every $b > 0$, a.s., the limit $\mathcal{A}_b := \lim_{\varepsilon \rightarrow 0} \mathcal{A}_b^\varepsilon$ exists and it is a linear mapping of rank $n - 1$. For any \mathbf{v}_0 , with probability 1, $\mathcal{A}_b^\varepsilon \mathbf{v}_0 \rightarrow \mathcal{A}_b \mathbf{v}_0$ as $\varepsilon \rightarrow 0$, uniformly in b on compact sets.

Recall the stochastic flow X_t^x of reflected Brownian motions defined in (1.3). By Theorem 3.1 of [3], for every $x \in D$, $b > 0$ and compact set $K \subset \mathbb{R}^n$, we have a.s.,

$$\limsup_{\varepsilon \rightarrow 0} \sup_{\mathbf{v} \in K} \left| (X_{\sigma_b^x}^{x+\varepsilon \mathbf{v}} - X_{\sigma_b^x}^x) / \varepsilon - \mathcal{A}_b \mathbf{v} \right| = 0, \quad (2.4)$$

where $\sigma_b^x = \inf\{t \geq 0 : L_t^{X^x} \geq b\}$.

Consider some $b > 0$ and let $\sigma_* = \inf\{t \geq 0 : L_t^X \vee L_t^Y \geq b\}$. Thus defined σ_* is different from the random variable with the same name in [3]. But it is easy to check that for any value of $b > 0$, one can choose parameters k_* and c_* in [3] (they are discussed after Corollary 3.2 in [3]), so that the present σ_* is less than the random variable with the same name in [3]. Since this random variable is used as an upper bound for some quantities in arguments in [3], we see that all results in [3] still hold with the definition of σ_* given in this paper.

For $\varepsilon_* > 0$, let

$$\left\{ e_{t_1^*}, e_{t_2^*}, \dots, e_{t_{m^*}^*} \right\} = \{e_t \in \mathcal{E}_b : |e_t(0) - e_t(\zeta -)| \geq \varepsilon_*, t < \sigma_*\}. \quad (2.5)$$

We label these excursions so that $t_k^* < t_{k+1}^*$ for all k and we let $\ell_k^* = L_{t_k^*}^X$ for $k = 1, \dots, m^*$. We also let $t_0^* = \inf\{t \geq 0 : X_t \in \partial D\}$, $\ell_0^* = 0$, $\ell_{m^*+1}^* = L_{\sigma_*}^X$, and $\Delta \ell_k^* = \ell_{k+1}^* - \ell_k^*$. Let $x_k^* = e_{t_k^*}(\zeta -)$ for $k = 1, \dots, m^*$, and $x_0^* = X_{t_0^*}$. Let

$$\mathcal{I}_k = \exp(\Delta \ell_k^* \mathcal{S}(x_k^*)) \pi_{x_k^*}.$$

The arguments in [3] were given only for $b = 1$ but it is easy to see that they apply equally to any fixed value of $b > 0$.

Fix an arbitrarily small $c_3 > 0$. By (3.161) and (3.167) of [3], there exist $c_4, c_5, c_6, \varepsilon_0 > 0$, $\beta_1 \in (1, 4/3)$ and $\beta_2 \in (0, 4/3 - \beta_1)$ such that if $X_0 = x$, $Y_0 = y$, $|x - y| = \varepsilon < \varepsilon_0$ and $\varepsilon_* = c_4 \varepsilon$ then

$$|(Y_{\sigma_*} - X_{\sigma_*}) - \mathcal{I}_{m^*} \circ \dots \circ \mathcal{I}_0(Y_0 - X_0)| \leq |\Lambda| + \Xi, \quad (2.6)$$

where $|\Lambda| < c_3 \varepsilon$, $\mathbb{P}^{x,y}$ -a.s., and

$$\mathbb{P}^{x,y}(|\Xi| > c_5 \varepsilon^{\beta_1}) \leq c_6 \varepsilon^{\beta_2}. \quad (2.7)$$

3. RECURRENCE OF SYNCHRONOUS COUPLINGS IN 3-DIMENSIONAL TORUS

Let \mathbb{P}^{x_0, y_0} denote the distribution of the solution (X, Y) to (1.1)-(1.2), and let \mathbb{E}^{x_0, y_0} denote the corresponding expectation. If no confusion may arise, x_0 and y_0 will be suppressed in the notation and we will use the notation “ \mathbb{P} ” and “a.s.”

Lemma 3.1. *Suppose that $x_1 \in \partial D$, $c_1 \in (0, 1/100)$ and let $D_2 = D \cap \mathcal{B}(x_1, c_1/4)$. Assume that $x_0, y_0 \in D_2$ and $|\langle x_0 - y_0, \mathbf{n}(x_1) \rangle| \leq c_1|x_0 - y_0|$. Let $T_1 = \tau_{D_2}^X \wedge \tau_{D_2}^Y$. Suppose that X and Y solve (1.1)-(1.2) with $X_0 = x_0$ and $Y_0 = y_0$. Then $|\langle X_t - Y_t, \mathbf{n}(x_1) \rangle| \leq c_1|X_t - Y_t|$ for all $t \leq T_1$.*

Proof. For $x_2 \in \partial D \cap D_2$ and $y_2 \in D_2$ we have $\langle x_2 - y_2, \mathbf{n}(x_1) \rangle \leq c_1|x_2 - y_2|/2$. For any $x_3 \in \partial D \cap D_2$, the angle between $\mathbf{n}(x_1)$ and $\mathbf{n}(x_3)$ is less than $c_1/2$ radians.

Assume that $|\langle X_t - Y_t, \mathbf{n}(x_1) \rangle| > c_1|X_t - Y_t|$ for some $t \leq T_1$. We will show that this assumption leads to a contradiction. Let

$$T_2 = \inf\{t \geq 0 : |\langle X_t - Y_t, \mathbf{n}(x_1) \rangle| > c_1|X_t - Y_t|\}.$$

By assumption, $T_2 < T_1$. We have $|\langle X_{T_2} - Y_{T_2}, \mathbf{n}(x_1) \rangle| = c_1|X_{T_2} - Y_{T_2}|$ so at most one of the points X_{T_2} and Y_{T_2} belongs to the boundary of D . At least one of these points belongs to ∂D because $t \rightarrow |\langle X_t - Y_t, \mathbf{n}(x_1) \rangle|/|X_t - Y_t|$ is constant over intervals where neither X nor Y visit ∂D . Suppose without loss of generality that $X_{T_2} \in \partial D$. Then, by the opening remarks, $\langle X_{T_2} - Y_{T_2}, \mathbf{n}(x_1) \rangle \leq c_1|X_{T_2} - Y_{T_2}|/2$, and therefore, $\langle X_{T_2} - Y_{T_2}, \mathbf{n}(x_1) \rangle = -c_1|X_{T_2} - Y_{T_2}|$. Since the angle between $\mathbf{n}(x_1)$ and $\mathbf{n}(X_{T_2})$ is less than $c_1/2$ radians, there exists a random time $T_3 < T_2$ such that $L_{T_2}^X - L_{T_3}^X > 0$, $L_{T_3}^Y - L_{T_2}^Y = 0$ and $\int_{T_3}^{T_2} \mathbf{n}(X_s) dL_s^X$ forms an angle less than c_1 with $\mathbf{n}(x_1)$. All these facts and the formula $X_{T_2} - Y_{T_2} = X_{T_3} - Y_{T_3} + \int_{T_3}^{T_2} \mathbf{n}(X_s) dL_s^X$ show that $\langle X_{T_3} - Y_{T_3}, \mathbf{n}(x_1) \rangle < -c_1|X_{T_3} - Y_{T_3}|$, contradicting the definition of T_2 . This completes the proof of the lemma. \square

Lemma 3.2. *If $x, y \in \overline{D}$ and $x \neq y$ then $\mathbb{P}^{x,y}(X_t \neq Y_t, \forall t \geq 0) = 1$.*

Proof. Assume that for some distinct $x, y \in \overline{D}$, $X_t = Y_t$ for some $t < \infty$, with positive probability. A standard application of the Markov property shows that there must exist $r \in (0, 1/100)$, $x_1 \in \partial D$ and $y_1 \in \overline{D}$ such that if we write $D_2 = D \cap \mathcal{B}(x_1, r/8)$ and $T_1 = \tau_{D_2}^X \wedge \tau_{D_2}^Y$ then $\mathbb{P}^{x_1, y_1}(\exists t \in [0, T_1] : X_t = Y_t) > 0$. Fix x_1, y_1 and r with these properties and let $K_\delta = \mathcal{T}_{x_1} \partial D \cap \partial \mathcal{B}(x_1, \delta)$. Recall the stochastic flow X_t^x of reflected Brownian motions defined in (1.3) and note that $(X_t, Y_t) = (X_t^{x_1}, X_t^{y_1})$ under \mathbb{P}^{x_1, y_1} . Let $\hat{\sigma}_b = \inf\{t \geq 0 : L_t^{X^{x_1}} \geq b\}$. According to Theorem 3.2 of [5] and its proof, for any fixed $b > 0$, \mathcal{A}_b has rank 2. In fact, the proof shows more than that, namely, \mathbb{P}^{x_1} -a.s., $\inf_{\mathbf{v} \in K_\delta} |\mathcal{A}_b(\mathbf{v})| > 0$. This and (2.4) imply that for any $b > 0$,

$$\lim_{\delta \rightarrow 0} \mathbb{P}^{x_1} \left(\inf_{\mathbf{v} \in K_\delta} |X_{\hat{\sigma}_b}^{x_1 + \mathbf{v}} - X_{\hat{\sigma}_b}^{x_1}| / |\mathbf{v}| > 0 \right) = 1.$$

Since the stochastic differential equation (1.1) has a unique strong solution, if $X_t^x = X_t^y$ for some t , then $X_s^x = X_s^y$ for all $s \geq t$, a.s. Hence, the last formula can be strengthened as

follows,

$$\lim_{\delta \rightarrow 0} \mathbb{P}^{x_1} \left(\inf_{\mathbf{v} \in K_\delta} \inf_{0 \leq t \leq \hat{\sigma}_b} |X_t^{x_1+\mathbf{v}} - X_t^{x_1}| / |\mathbf{v}| > 0 \right) = 1.$$

For every $k \geq 1$ find $\delta_k > 0$ such that

$$\mathbb{P}^{x_1} \left(\inf_{\mathbf{v} \in K_{\delta_k}} \inf_{0 \leq t \leq \hat{\sigma}_b} |X_t^{x_1+\mathbf{v}} - X_t^{x_1}| / |\mathbf{v}| > 0 \right) \geq 1 - 2^{-k}. \quad (3.1)$$

It follows from Lemmas 3.3 and 3.4 of [3] and their proofs that there exist stopping times S_k such that $S_k \rightarrow \infty$ as $k \rightarrow \infty$, and $|X_t^x - X_t^y| \leq k|X_0^x - X_0^y|$ for all $x, y \in \overline{D}$ and $t \in [0, S_k]$, a.s. We can assume without loss of generality that $\delta_k \rightarrow 0$ as $k \rightarrow \infty$. We make $\delta_k > 0$ smaller, if necessary, so that $|X_t^{x_1} - X_t^y| \leq r/8$, for all $y \in K_{\delta_k}$ and $t \in [0, S_k]$, a.s. By passing to a subsequence, if necessary, we may assume that

$$\mathbb{P}(S_k > \hat{\sigma}_b) \geq 1 - 2^{-k}. \quad (3.2)$$

If we let

$$\begin{aligned} F_k^1 &= \{|X_{\hat{\sigma}_b}^{x_1} - X_{\hat{\sigma}_b}^y| \leq r/8, \quad \forall y \in K_{\delta_k}, t \in [0, \tilde{\sigma}_b]\}, \\ F_k^2 &= \left\{ \inf_{\mathbf{v} \in K_{\delta_k}} \inf_{0 \leq t \leq \hat{\sigma}_b} |X_t^{x_1+\mathbf{v}} - X_t^{x_1}| / |\mathbf{v}| > 0 \right\}, \\ F_k &= F_k^1 \cap F_k^2, \end{aligned}$$

then, by (3.1) and (3.2), $\mathbb{P}(F_k) \geq 1 - 2^{-k+1}$.

Consider the case when $|\langle x_1 - y_1, \mathbf{n}(x_1) \rangle| \leq (r/8)|x_1 - y_1|$. Let k_0 be so large that for $k \geq k_0$, $|\langle x_1 - z, \mathbf{n}(x_1) \rangle| \leq (r/4)|x_1 - z|$ and $|\langle z - y_1, \mathbf{n}(x_1) \rangle| \leq (r/4)|z - y_1|$ for all $z \in K_{\delta_k}$. Let $T_z^2 = T_{\partial D}^{X_z} \wedge T_{\partial D}^Y$. Suppose that $T_z^2 = T_{\partial D}^{X_z}$. Then $|\langle X_{T_z^2}^z - Y_{T_z^2}, \mathbf{n}(X_{T_z^2}^z) \rangle| \leq (r/2)|X_{T_z^2}^z - Y_{T_z^2}|$. If F_k^1 holds then $X_t^z \in \mathcal{B}(x_1, r/4)$ for all $t \in [0, \tilde{\sigma}_b]$ and $z \in K_{\delta_k}$. Hence, we can apply Lemma 3.1 to see that $|\langle X_t^z - Y_t, \mathbf{n}(X_{T_z^2}^z) \rangle| \leq (r/2)|X_t^z - Y_t|$ for all $t \in [T_z^2, \tilde{\sigma}_b]$. A similar argument applies in the case $T_z^2 = T_{\partial D}^Y$ and gives $|\langle X_t^z - Y_t, \mathbf{n}(Y_{T_z^2}) \rangle| \leq (r/2)|X_t^z - Y_t|$. The claim holds for all $z \in K_{\delta_k}$ simultaneously because Lemma 3.1 is deterministic. Similarly, $|\langle X_t^{x_1} - X_t^z, \mathbf{n}(x_1) \rangle| \leq (r/2)|X_t^{x_1} - X_t^z|$ for all $t \in [T_z^2, \tilde{\sigma}_b]$.

The above estimates have the following topological interpretation. Recall that $\pi_{x_1} \mathbf{z}$ denotes the projection of \mathbf{z} on $\mathcal{T}_{x_1} \partial D$. Assuming that F_k^2 holds and t is fixed, the set $\Gamma_t = \pi_{x_1} \{X_s^x, x \in K_{\delta_k}\}$ is a closed loop that contains $\pi_{x_1} X_t$ inside. When t goes from 0 to $\tilde{\sigma}_b$, $\pi_{x_1} X_t$, $\pi_{x_1} Y_t$ and Γ_t evolve continuously. If $X_t = Y_t$ for some t then we must have $\pi_{x_1} Y_s = \pi_{x_1} X_s^x$ for some $x \in K_{\delta_k}$ and $0 \leq s \leq t$, and, therefore, $X_t = X_t^x$. But this means that F_k^2 does not hold. Since $\mathbb{P}(F_k) \geq 1 - 2^{-k+1}$, we conclude that the probability that there exists $t \in [0, \hat{\sigma}_b]$ such that $X_t = Y_t$ is less than 2^{-k+1} . Since k and b are arbitrarily large, $\mathbb{P}^{x_1, y_1}(\exists t \in [0, T_1] : X_t = Y_t) = 0$.

Next consider the case of arbitrary $X_0 = x_1 \in \partial D$ and $Y_0 = y_1 \in \overline{D}_2$. Assume that $\mathbb{P}^{x_1, y_1}(\exists t \in [0, T_1] : Y_t = X_t) = p_1 > 0$. We will show that this assumption leads to a contradiction. Let $A = \{y \in \overline{D} : |x_1 - y| = |x_1 - y_1|, \langle x_1 - y, \mathbf{n}(x_1) \rangle = \langle x_1 - y, \mathbf{n}(x_1) \rangle\}$. The set A is a circle, possibly with a zero radius. If the radius of A is 0, that is, if A contains only y_1 , then $x_1 - y_1$ is parallel to $\mathbf{n}(x_1)$. It is easy to see that with probability 1, there exists time $t \leq T_1$ such that $X_t \neq Y_t$, $X_t \in \partial D$, and $X_t - Y_t$ is not parallel to $\mathbf{n}(X_t)$. Let U_k be the smallest such t greater than 2^{-k} . We can apply the strong Markov property at time U_k , for every k , and the result proved below for the case when A does not reduce to a single point.

Hence, we will assume from now on that the set A is a circle with a non-zero radius. Consider any $y_1, y_2 \in A$ and reflected Brownian motions X^{y_1} and X^{y_2} . We will show that

$$\mathbb{P} \left(\inf_{0 \leq t \leq T_1} |X_t^{y_1} - X_t^{y_2}| > 0 \right) = 1. \quad (3.3)$$

If the processes X^{y_1} and X^{y_2} do not hit ∂D before T_1 then of course they do not meet before T_1 . If they hit the boundary of D then we can suppose without loss of generality that $T_3 := T_{\partial D}^{X^{y_1}} \leq T_{\partial D}^{X^{y_2}}$. Then $|\langle X_{T_3} - Y_{T_3}, \mathbf{n}(X_{T_3}) \rangle| \leq (r/8)|X_{T_3} - Y_{T_3}|$. We see that (3.3) holds by the first part of the proof and the strong Markov property applied at T_3 . Since we have assumed that $\mathbb{P}^{x_1, y_1}(\exists t \in [0, T_1] : |Y_t - X_t| = 0) = p_1$, we have

$$\mathbb{P}^{x_1, y_1}(Y_{T_1} = X_{T_1}) = p_1, \quad (3.4)$$

because once any two solutions meet, they have to stay identical forever, by the strong uniqueness. Now choose n distinct points y_1, \dots, y_n in A , with $n > 2/p_1$. By (3.4), symmetry, and (3.3) we have

$$\begin{aligned} \mathbb{P}(X_{T_1}^{x_1} = X_{T_1}^{y_j}) &= p_1, & j &= 1, \dots, n, \\ \mathbb{P}(X_{T_1}^{y_i} = X_{T_1}^{y_j}) &= 0, & i &\neq j. \end{aligned}$$

This is a contradiction. The proof is complete. \square

Lemma 3.3. *For any $b > 0$ and $\beta_1 \in (0, 1)$ there exist $c_0, \beta_2, \varepsilon_0 > 0$ such that if $\varepsilon \leq \varepsilon_0$, $x, y \in \overline{D}$ and $|x - y| = \varepsilon$ then*

$$\mathbb{P}^{x, y} \left(\frac{\left| \langle Y_{\sigma_b^x} - X_{\sigma_b^x}, \mathbf{n}(X_{\sigma_b^x}) \rangle \right|}{|Y_{\sigma_b^x} - X_{\sigma_b^x}|} \geq c_0 \varepsilon^{\beta_1} \right) \leq \varepsilon^{\beta_2}. \quad (3.5)$$

Proof. The proof is similar to the proof of Lemma 4.6 in [4] so we only sketch the main ideas. The paper [4] is concerned with 2-dimensional domains but it is easy to see that the results from that paper that we use here apply to multidimensional domains.

By Lemma 4.1 (ii) of [4], $\mathbb{P}(L_{\sigma_b^x}^Y \geq a) \leq c_1 e^{-c_2 a}$. Hence, for any $\beta_3 > 0$ and some $\beta_4 > 0$,

$$\mathbb{P}(L_{\sigma_b^x}^Y \geq \beta_3 |\log \varepsilon|) \leq c_1 \exp(-c_2 \beta_3 |\log \varepsilon|) = c_1 \varepsilon^{\beta_4}.$$

If the event $A_1 := \{L_{\sigma_b^X}^Y \leq \beta_3 |\log \varepsilon|\}$ holds then, by Lemma 3.8 of [4],

$$\sup_{t \in [0, \sigma_b^X]} |X_t - Y_t| \leq |X_0 - Y_0| \exp(c_4(1 + \beta_3 |\log \varepsilon|)) \leq c_5 \varepsilon^{1-c_4\beta_3} = c_5 \varepsilon^{1-\beta_5},$$

where β_5 is defined as $c_4\beta_3$. Choose $\beta_3 > 0$ so small that we can find β_1 and β_6 such that $\beta_5 < \beta_1 < \beta_6 < 1 - \beta_5$.

Let $T_1 = \inf\{t \geq 0 : X_t \in \partial D\}$ and $\{V_t, 0 \leq t \leq \sigma_b^X - T_1\} := \{X_{\sigma_b^X - t}, 0 \leq t \leq \sigma_b^X - T_1\}$. If we condition on the values of X_{T_1} and $X_{\sigma_b^X}$, the process V is a reflected Brownian motion in D starting from $X_{\sigma_b^X}$ and conditioned to approach X_{T_1} at its lifetime. It is easy to see that $\mathbb{P}(|X_{T_1} - X_{\sigma_b^X}| \leq \varepsilon^{\beta_1}) \leq c_6 \varepsilon^{\beta_1}$.

Suppose that the event $A_2 := \{\text{dist}(X_{T_1}, X_{\sigma_b^X}) \geq \varepsilon^{\beta_1}\}$ holds. Conditional on this event, the probability that V does not spend ε^{β_6} units of local time on the boundary of ∂D before leaving the disc $\mathcal{B}(V_0, \varepsilon^{\beta_1})$ is bounded by $c_7 \varepsilon^{\beta_6 - \beta_1}$. Let A_3 be the event that V spends ε^{β_6} or more units of local time on the boundary of ∂D before leaving the disc $\mathcal{B}(V_0, \varepsilon^{\beta_1})$. Let $T_2 = \sup\{t \leq \sigma_b^X : X_t \notin \mathcal{B}(V_0, \varepsilon^{\beta_1})\}$. If A_3 holds then Y must hit ∂D at some time $t \in [T_2, \sigma_b^X]$ because $\varepsilon^{\beta_6} > c_5 \varepsilon^{1-\beta_5}$ for small ε , i.e. the amount of push given to Y exceeds the maximum distance between the two processes. We also have $X_{\sigma_b^X} \in \partial D$. The maximum angle between normal vectors at points of $\partial D \cap \mathcal{B}(V_0, \varepsilon^{\beta_1})$ is less than $c_8 \varepsilon^{\beta_1}$. A modification of Lemma 3.1 shows that $|\langle X_{\sigma_b^X} - Y_{\sigma_b^X}, \mathbf{n}(X_{\sigma_b^X}) \rangle| \leq |X_{\sigma_b^X} - Y_{\sigma_b^X}| c_9 \varepsilon^{\beta_1}$. Recall that, by Lemma 3.2, $|X_{\sigma_b^X} - Y_{\sigma_b^X}| > 0$, a.s. We have shown that (3.5) holds true if $A_1 \cap A_2 \cap A_3$ holds. Since $\mathbb{P}((A_1 \cap A_2 \cap A_3)^c) \leq \varepsilon^{\beta_4} + c_6 \varepsilon^{\beta_1} + c_7 \varepsilon^{\beta_6 - \beta_1}$, the lemma follows. \square

Let $\sigma_t^X = \inf\{s \geq 0 : L_s^X \geq t\}$, $\sigma_t^Y = \inf\{s \geq 0 : L_s^Y \geq t\}$ and $\sigma_b^* = \sigma_b^X \wedge \sigma_b^Y$. The random variable σ_b^* was denoted σ_* in Section 2.4 for consistency with the notation of [3]. The new notation, σ_b^* , is more appropriate for this paper. An alternative formula is $\sigma_b^* = \inf\{t \geq 0 : L_t^X \vee L_t^Y \geq b\}$. Hence, according to this convention, one would expect that $\sigma_{kb}^* = \inf\{t \geq 0 : L_t^X \vee L_t^Y \geq kb\}$. It will be convenient to give a different meaning to σ_{kb}^* , namely, $\sigma_{(k+1)b}^* = \inf\left\{t \geq \sigma_{kb}^* : (L_t^X - L_{\sigma_{kb}^*}^X) \vee (L_t^Y - L_{\sigma_{kb}^*}^Y) \geq b\right\}$ for $k \geq 1$. Let

$$\begin{aligned} R_t &= |X_t - Y_t|, & M_t &= \log R_t, & t &\geq 0, \\ V_k &= M_{\sigma_{kb}^*}, & k &= 0, 1, \dots \end{aligned} \tag{3.6}$$

Lemma 3.4. *For any $\beta_1 \in (0, 1)$ and $p < 1$ there exist $\beta_2, c_1, b, \varepsilon_1 > 0$ such that if $\varepsilon \leq \varepsilon_1$, $x_0 \in \partial D$, $y_0 \in \overline{D}$, $|x_0 - y_0| = \varepsilon$, $X_0 = x_0$, $Y_0 = y_0$ and*

$$\frac{|\langle y_0 - x_0, \mathbf{n}(x_0) \rangle|}{|y_0 - x_0|} \leq \varepsilon^{\beta_1} \tag{3.7}$$

then

$$\mathbb{P}^{x_0, y_0}(V_1 - V_0 \geq c_1) \geq p. \tag{3.8}$$

Proof. Step 1. Recall the results from [3] reviewed in Section 2.4. Suppose that $\varepsilon_* > 0$, $x_0 \in \partial D$, $\mathbf{v} \in \mathcal{T}_{x_0} \partial D$, $|\mathbf{v}| = 1$, $X_0 = x_0$ and let e_u be the first excursion of X from ∂D with $|e_u(0) - e_u(\zeta-)| \geq \varepsilon_*$. Let $x_1 = e_u(\zeta-)$ and $\alpha = 3/4$. We will estimate $\mathbb{P}^{x_0}(|x_0 - e_u(0)| \geq \varepsilon_*^\alpha)$ and $\mathbb{E}^{x_0}(|\log |\pi_{x_1} \mathbf{v}|| \mathbf{1}_{\{|x_0 - e_u(0)| \leq \varepsilon_*^\alpha\}})$.

Let $U_1 = \mathcal{B}(x_0, \varepsilon_*) \cap \partial D$, $U_k = (\mathcal{B}(x_0, k\varepsilon_*) \setminus \mathcal{B}(x_0, (k-1)\varepsilon_*)) \cap \partial D$ for $k \geq 2$, and $T_0 = 0$. Let j_k be such that $T_k \in U_{j_k}$, and $T_k = \inf\{t \geq T_{k-1} : X_t \in \partial D \setminus (U_{j_{k-1}} \cup U_{j_k} \cup U_{j_{k+1}})\}$ for $k \geq 1$.

Recall that u denotes the starting time of the first excursion of X from ∂D with $|e_u(0) - e_u(\zeta-)| \geq \varepsilon_*$. Let p_1 be the probability that $|x_0 - e_u(0)| < \varepsilon_*$ and note that $p_1 > 0$. The strong Markov property applied at T_k shows that $\mathbb{P}^{x_0}(u \leq T_{k+1} \mid u \geq T_k) \geq p_1$. It follows that $\mathbb{P}^{x_0}(u \geq T_k) \leq (1 - p_1)^k$. For the event $\{|x_0 - e(0)| \geq \varepsilon_*^\alpha\}$, we have to have $u \geq T_k$ with $k \geq \varepsilon_*^\alpha / (2\varepsilon_*)$. It follows that, setting $c_1 = -(1/2) \log(1 - p_1) > 0$,

$$\mathbb{P}^{x_0}(|x_0 - e_u(0)| \geq \varepsilon_*^\alpha) \leq (1 - p_1)^{\varepsilon_*^\alpha / (2\varepsilon_*)} = \exp(-c_1 \varepsilon_*^{\alpha-1}). \quad (3.9)$$

Let $\beta = 5/8$ and note that if $|x_1 - x_0| \leq \varepsilon_*^\beta$ then $|\log |\pi_{x_1} \mathbf{v}|| \leq c_2 \varepsilon_*^{2\beta}$. Hence,

$$\begin{aligned} \mathbb{E}^{x_0}(|\log |\pi_{x_1} \mathbf{v}|| \mathbf{1}_{\{|x_0 - e_u(0)| \leq \varepsilon_*^\alpha\}}) &= \mathbb{E}^{x_0} \left(|\log |\pi_{x_1} \mathbf{v}|| \mathbf{1}_{\{|x_0 - e_u(0)| \leq \varepsilon_*^\alpha\}} \mathbf{1}_{\{|x_1 - x_0| \leq \varepsilon_*^\beta\}} \right) \\ &\quad + \mathbb{E}^{x_0} \left(|\log |\pi_{x_1} \mathbf{v}|| \mathbf{1}_{\{|x_0 - e_u(0)| \leq \varepsilon_*^\alpha\}} \mathbf{1}_{\{|x_1 - x_0| \geq \varepsilon_*^\beta\}} \right) \\ &\leq c_2 \varepsilon_*^{2\beta} + \mathbb{E}^{x_0} \left(|\log |\pi_{x_1} \mathbf{v}|| \mathbf{1}_{\{|x_0 - e_u(0)| \leq \varepsilon_*^\alpha\}} \mathbf{1}_{\{|x_1 - x_0| \geq \varepsilon_*^\beta\}} \right). \end{aligned} \quad (3.10)$$

It follows from (4.1) and (4.8) that for large ρ , small ε_* , $|x_1 - x_0| \geq \varepsilon_*^\beta$ and $|x_0 - x| \leq \varepsilon_*^\alpha$,

$$\frac{H^x(e_{\zeta-} \in dx_1)}{H^{x_0}(e_{\zeta-} \in dx_1)} \leq \frac{|x - x_1|^{-3}}{|x_0 - x_1|^{-3}} \leq \frac{(\varepsilon_*^\beta - \varepsilon_*^\alpha)^{-3}}{\varepsilon_*^{-3\beta}} \leq 1 + 6\varepsilon_*^{\alpha-\beta}. \quad (3.11)$$

Let $c_* = \sqrt{2} + \log 2 - 2 - \log(1 + \sqrt{2}) \approx -0.77$ be the constant in the statement of Theorem 4.1 (ii). Theorem 4.1 (ii), the exit system formula (2.1) and (3.11) imply that for arbitrarily small $c_3 > 0$, any $c_4 \in (-c_*, 1)$, large ρ and small ε_* ,

$$\begin{aligned} &\mathbb{E}^{x_0} \left(|\log |\pi_{x_1} \mathbf{v}|| \mathbf{1}_{\{|x_0 - e_u(0)| \leq \varepsilon_*^\alpha\}} \mathbf{1}_{\{|x_1 - x_0| \geq \varepsilon_*^\beta\}} \right) \\ &= \mathbb{E}^{x_0} \frac{H^{X_u} \left(|\log |\pi_{e(\zeta-)} \mathbf{v}|| \mathbf{1}_{\{|x_0 - X_u| \leq \varepsilon_*^\alpha\}} \mathbf{1}_{\{|e(\zeta-) - x_0| \geq \varepsilon_*^\beta\}} \right)}{H^{X_u}(\mathbf{1}_{\{|e(\zeta-) - X_u| \geq \varepsilon_*\}})} \\ &\leq \frac{(1 + 6\varepsilon_*^{\alpha-\beta}) H^{x_0} \left(|\log |\pi_{e(\zeta-)} \mathbf{v}|| \mathbf{1}_{\{|e(\zeta-) - x_0| \geq \varepsilon_*^\beta\}} \right)}{H^{x_0}(\mathbf{1}_{\{|e(\zeta-) - x_0| \geq \varepsilon_*\}})} \\ &\leq \frac{(1 + 6\varepsilon_*^{\alpha-\beta})(c_3 + \sqrt{2} + \log 2 - 2 - \log(1 + \sqrt{2}))}{H^{x_0}(\mathbf{1}_{\{|e(\zeta-) - x_0| \geq \varepsilon_*\}})} \\ &\leq c_4 / H^{x_0}(\mathbf{1}_{\{|e(\zeta-) - x_0| \geq \varepsilon_*\}}). \end{aligned}$$

We combine the last estimate and (3.10) to obtain

$$\mathbb{E}^{x_0} \left(|\log |\pi_{x_1} \mathbf{v}|| \mathbf{1}_{\{|x_0 - e_u(0)| \leq \varepsilon_*^\alpha\}} \right) \leq c_2 \varepsilon_*^{2\beta} + c_4 / H^{x_0} \left(\mathbf{1}_{\{|e(\zeta-) - x_0| \geq \varepsilon_*\}} \right). \quad (3.12)$$

Recall the notation from the paragraph containing (2.5). Consider an arbitrary $\mathbf{v}_0 \in \mathbb{R}^3$. Since ∂D is a sphere with the unit radius, $\mathcal{S}(x)$ is the identity operator so $\mathcal{I}_k = \exp(\Delta \ell_k^*) \pi_{x_k^*}$ and, therefore,

$$\begin{aligned} \mathcal{I}_{m^*} \circ \cdots \circ \mathcal{I}_0(\mathbf{v}_0) &= \exp \left(\sum_{0 \leq k \leq m^*} \Delta \ell_k^* \right) \pi_{x_{m^*}^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0) \\ &= \exp(\ell_{m^*+1}^*) \pi_{x_{m^*}^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0) \\ &= \exp(L_{\sigma_*}) \pi_{x_{m^*}^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0). \end{aligned} \quad (3.13)$$

We want to find a lower bound for the above expression. Since $\exp(L_{\sigma_*}) \geq 1$, it will suffice to analyze the composition of projection operators. We have

$$\begin{aligned} &\log \left| \pi_{x_{m^*}^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0) \right| \\ &= \sum_{1 \leq k \leq m^*} \left(\log \left| \pi_{x_k^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0) \right| - \log \left| \pi_{x_{k-1}^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0) \right| \right) + \log \left| \pi_{x_0^*}(\mathbf{v}_0) \right|. \end{aligned} \quad (3.14)$$

By the strong Markov property applied at the excursion endpoint $s_{k-1} := u_{k-1} + e_{u_{k-1}}(\zeta-)$, the conditional distribution of

$$\log \left| \pi_{x_k^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0) \right| - \log \left| \pi_{x_{k-1}^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0) \right|$$

given $\mathcal{F}_{s_{k-1}}$ is the same as that of $|\log |\pi_{x_1} \mathbf{v}||$, introduced at the beginning of the proof. Let

$$F_k = \left\{ \log \left| \pi_{x_k^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0) \right| - \log \left| \pi_{x_{k-1}^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0) \right| \leq \varepsilon_*^\alpha \right\}.$$

We see that events F_k , $k \geq 1$, are independent and so are random variables

$$\left| \log \left| \pi_{x_k^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0) \right| - \log \left| \pi_{x_{k-1}^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0) \right| \right| \mathbf{1}_{F_k}. \quad (3.15)$$

It follows from (3.12) that

$$\begin{aligned} &\mathbb{E}^{z_0} \left(\left| \log \left| \pi_{x_k^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0) \right| - \log \left| \pi_{x_{k-1}^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0) \right| \right| \mathbf{1}_{F_k} \right) \\ &\leq c_2 \varepsilon_*^{2\beta} + c_4 / H^{x_0} \left(\mathbf{1}_{\{|e(\zeta-) - x_0| \geq \varepsilon_*\}} \right). \end{aligned}$$

Thus the process

$$\begin{aligned} N_n &= n(c_2 \varepsilon_*^{2\beta} + c_4 / H^{x_0} \left(\mathbf{1}_{\{|e(\zeta-) - x_0| \geq \varepsilon_*\}} \right)) \\ &\quad - \sum_{1 \leq k \leq n} \left| \log \left| \pi_{x_k^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0) \right| - \log \left| \pi_{x_{k-1}^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0) \right| \right| \mathbf{1}_{F_k} \end{aligned}$$

is a submartingale. By the optional stopping theorem, $\mathbb{E}^{z_0} N_{m^*} \geq 0$, so

$$\begin{aligned} & \mathbb{E}^{z_0} \left(\sum_{1 \leq k \leq m^*} \left| \log |\pi_{x_k^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0)| - \log |\pi_{x_{k-1}^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0)| \right| \mathbf{1}_{F_k} \right) \\ & \leq \mathbb{E}^{z_0} m^* (c_2 \varepsilon_*^{2\beta} + c_4 / H^{x_0} (\mathbf{1}_{\{|e(\zeta-) - x_0| \geq \varepsilon_*\}})). \end{aligned} \quad (3.16)$$

It is easy to see that $H^{x_0} (\mathbf{1}_{\{|e(\zeta-) - x_0| \geq \varepsilon_*\}}) \leq c_5 / \varepsilon_*$. It follows from the definition of m^* and the exit system formula (2.1) that m^* has the Poisson distribution with the expected value $bH^{x_0} (\mathbf{1}_{\{|e(\zeta-) - x_0| \geq \varepsilon_*\}})$. These observations and (3.16) yield for some $c_6 > 0$, any $c_7 \in (c_3, 1)$ and small ε_* ,

$$\begin{aligned} & \mathbb{E}^{z_0} \left(\sum_{1 \leq k \leq m^*} \left| \log |\pi_{x_k^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0)| - \log |\pi_{x_{k-1}^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0)| \right| \prod_{1 \leq j \leq m^*} \mathbf{1}_{F_j} \right) \\ & \leq \mathbb{E}^{z_0} \left(\sum_{1 \leq k \leq m^*} \left| \log |\pi_{x_k^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0)| - \log |\pi_{x_{k-1}^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0)| \right| \mathbf{1}_{F_k} \right) \\ & \leq (c_6 \varepsilon_*^{2\beta-1} + c_4) b \leq c_7 b. \end{aligned} \quad (3.17)$$

In addition, since we are dealing with a sum of i.i.d. random variables given in (3.15), and the sum has a Poisson number m^* of terms with large mean, it is easy to see that for any $c_8 \in (c_7, 1)$ and $p_2 > 0$ there exist b_1 and ε_0 such that for $b \geq b_1$ and $\varepsilon_* \leq \varepsilon_0$,

$$\mathbb{P}^{z_0} \left(\sum_{1 \leq k \leq m^*} \left| \log |\pi_{x_k^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0)| - \log |\pi_{x_{k-1}^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0)| \right| \mathbf{1}_{F_k} \geq c_8 b \right) \leq p_2. \quad (3.18)$$

A similar argument based on the strong Markov property applied at times s_k and the optional stopping theorem for submartingales, combined with (3.9), gives

$$\mathbb{P}^{z_0} \left(\bigcup_{1 \leq k \leq m^*} F_k^c \right) \leq \mathbb{E}^{z_0} m^* \exp(-c_1 \varepsilon_*^{\alpha-1}) \leq c_9 \exp(-c_1 \varepsilon_*^{\alpha-1}) \varepsilon_*^{-1}. \quad (3.19)$$

Step 2. Recall the notation from Section 2.4. We copy below (2.6)-(2.7) because these estimates are crucial to the present argument. Fix an arbitrarily small $c_{10} > 0$. There exist $c_{11}, c_{12}, c_{13}, \varepsilon_0 > 0$, $\beta_1 \in (1, 4/3)$ and $\beta_2 \in (0, 4/3 - \beta_1)$ such that if $X_0 = x$, $Y_0 = y$, $|x - y| = \varepsilon < \varepsilon_0$ and $\varepsilon_* = c_{11}\varepsilon$ then

$$|(Y_{\sigma_b^*} - X_{\sigma_b^*}) - \mathcal{I}_{m^*} \circ \cdots \circ \mathcal{I}_0(Y_0 - X_0)| \leq |\Lambda| + \Xi, \quad (3.20)$$

where $|\Lambda| < c_{10}\varepsilon$, $\mathbb{P}^{x,y}$ -a.s., and

$$\mathbb{P}^{x,y}(|\Xi| > c_{12}\varepsilon^{\beta_1}) \leq c_{13}\varepsilon^{\beta_2}. \quad (3.21)$$

We have

$$\begin{aligned} & \log \left| \pi_{x_{m^*}}^* \circ \cdots \circ \pi_{x_0}^*(Y_0 - X_0) \right| \\ &= \sum_{1 \leq k \leq m^*} \left(\log \left| \pi_{x_k}^* \circ \cdots \circ \pi_{x_0}^*(Y_0 - X_0) \right| - \log \left| \pi_{x_{k-1}}^* \circ \cdots \circ \pi_{x_0}^*(Y_0 - X_0) \right| \right) \\ & \quad + \log \left| \pi_{x_0}^*(Y_0 - X_0) \right|. \end{aligned} \quad (3.22)$$

It follows from (3.7) that

$$\log \varepsilon - \log \left| \pi_{x_0}^*(Y_0 - X_0) \right| = \log \varepsilon - \log \left| \pi_{x_0}(y_0 - x_0) \right| \leq c_{14} \varepsilon^{2\beta_1}. \quad (3.23)$$

We combine this with (3.18), (3.19) and (3.22) to see that for any $c_{15} \in (c_7, 1)$ and $p_2 > 0$ there exist b_2 and $\varepsilon_1 > 0$ such that for $b \geq b_2$ and $\varepsilon \leq \varepsilon_1$,

$$\mathbb{P} \left(\left| \log \left| \pi_{x_{m^*}}^* \circ \cdots \circ \pi_{x_0}^*(Y_0 - X_0) \right| \right| \geq c_{15} b \right) \leq p_2. \quad (3.24)$$

A special case of (3.13) is

$$\mathcal{I}_{m^*} \circ \cdots \circ \mathcal{I}_0(Y_0 - X_0) = \exp \left(L_{\sigma_b^*} \right) \pi_{x_{m^*}}^* \circ \cdots \circ \pi_{x_0}^*(Y_0 - X_0).$$

This implies that

$$\log \left| \mathcal{I}_{m^*} \circ \cdots \circ \mathcal{I}_0(Y_0 - X_0) \right| = L_{\sigma_b^*} + \log \left| \pi_{x_{m^*}}^* \circ \cdots \circ \pi_{x_0}^*(Y_0 - X_0) \right|.$$

On the event $\{\sigma_b^* = \sigma_b^X\}$, we have $L_{\sigma_b^*} = b$ so if $|x_0 - y_0| = \varepsilon$ then, in view of (3.23) and (3.24),

$$\begin{aligned} & \mathbb{P}^{x_0, y_0} \left(\log \left| \mathcal{I}_{m^*} \circ \cdots \circ \mathcal{I}_0(Y_0 - X_0) \right| - b - \log \varepsilon \leq -c_{15} b \text{ and } \sigma_b^* = \sigma_b^X \right) \\ &= \mathbb{P}^{x_0, y_0} \left(\log \left| \mathcal{I}_{m^*} \circ \cdots \circ \mathcal{I}_0(Y_0 - X_0) \right| \leq (1 - c_{15})b + \log \varepsilon \text{ and } \sigma_b^* = \sigma_b^X \right) \leq p_2. \end{aligned} \quad (3.25)$$

Let $c_{16} = 1 - c_{15} > 0$. If $\sigma_b^* = \sigma_b^X$ and the event in (3.25) does not hold then

$$\left| \mathcal{I}_{m^*} \circ \cdots \circ \mathcal{I}_0(Y_0 - X_0) \right| \geq \varepsilon \exp(c_{16}b).$$

Recall from (3.20) that we can assume that $|\Lambda| \leq c_{10}\varepsilon$, a.s. Therefore,

$$\left| \mathcal{I}_{m^*} \circ \cdots \circ \mathcal{I}_0(Y_0 - X_0) \right| - |\Lambda| \geq \varepsilon(\exp(c_{16}b) - c_{10}). \quad (3.26)$$

It follows from (3.21) that for small ε , $\mathbb{P}(|\Xi| \geq c_{10}\varepsilon) < p_2$. This, (3.25) and (3.26) imply that

$$\mathbb{P}^{x, y} \left(\left| \mathcal{I}_{m^*} \circ \cdots \circ \mathcal{I}_0(Y_0 - X_0) \right| - |\Lambda| - |\Xi| \leq \varepsilon(\exp(c_{16}b) - 2c_{10}) \text{ and } \sigma_b^* = \sigma_b^X \right) \leq 2p_2.$$

We combine this estimate with (3.20) to see that

$$\begin{aligned} & \mathbb{P}^{x_0, y_0} \left(\left| (Y_{\sigma_b^X} - X_{\sigma_b^X}) \right| \leq \varepsilon(\exp(c_{16}b) - 2c_{10}) \text{ and } \sigma_b^* = \sigma_b^X \right) \\ &= \mathbb{P}^{x_0, y_0} \left(\left| (Y_{\sigma_b^*} - X_{\sigma_b^*}) \right| \leq \varepsilon(\exp(c_{16}b) - 2c_{10}) \text{ and } \sigma_b^* = \sigma_b^X \right) \leq 2p_2. \end{aligned} \quad (3.27)$$

We choose large b_3 so that for $b \geq b_3$, $c_{17} = c_{17}(b) := \exp(c_{16}b) - 3c_{10} > 1$. Let $c_{18} = \log c_{17} > 0$. Recall that $x_0 \in \partial D$, $y_0 \in \overline{D}$, and let $T' = \inf\{t \geq 0 : |X_t| = |Y_t|\}$. Note that the

distribution of $\{X_t, t \geq T'\}$ is the same as that of $\{Y_t, t \geq T'\}$. Moreover, $Y_t \notin \partial D$ for $t < T'$ and, therefore, $L_{T'}^Y = 0$. It follows that $\mathbb{P}^{x_0, y_0}(\sigma_b^* = \sigma_b^X) \geq 1/2$. This and (3.27) imply that

$$\begin{aligned} \mathbb{P}^{x_0, y_0}(V_1 - V_0 \leq c_{18}) &= \mathbb{P}^{x_0, y_0}\left(\log\left|(Y_{\sigma_b^X} - X_{\sigma_b^X})\right| - \log \varepsilon \leq \log c_{17}\right) \\ &= \mathbb{P}^{x_0, y_0}\left(\left|(Y_{\sigma_b^X} - X_{\sigma_b^X})\right| \leq \varepsilon(\exp(c_{16}b) - 2c_{10})\right) \\ &\leq 4p_2. \end{aligned} \quad (3.28)$$

The lemma is proved. \square

Recall notation from (3.6).

Lemma 3.5. *For any $\beta_1 \in (0, 1/2)$ there exist $\beta_2, b, c_1, \varepsilon_1 > 0$ and a cumulative distribution function $G : \mathbb{R} \rightarrow [0, 1]$ satisfying $\int_{-\infty}^{\infty} |a| dG(a) < \infty$ and such that if $\varepsilon \leq \varepsilon_1$, $x_0 \in \partial D$, $y_0 \in \overline{D}$, $|x_0 - y_0| = \varepsilon$, $X_0 = x_0$, $Y_0 = y_0$ and*

$$\frac{|\langle y_0 - x_0, \mathbf{n}(x_0) \rangle|}{|y_0 - x_0|} \leq \varepsilon^{\beta_1} \quad (3.29)$$

then there exists an event F such that

$$\mathbb{P}^{x_0, y_0}(F^c) \leq c_1 \varepsilon^{\beta_2}, \quad (3.30)$$

$$\mathbb{P}^{x_0, y_0}(|V_1 - V_0| \mathbf{1}_F \leq a) \leq G(a), \quad a \in \mathbb{R}. \quad (3.31)$$

Proof. Step 1. It follows from (3.27) that for some $p_1, \varepsilon_1 > 0$ and $c_2 = c_2(b)$, assuming that $|x_0 - y_0| = \varepsilon \leq \varepsilon_1$, and (3.29) holds,

$$\mathbb{P}^{x_0, y_0}(|Y_{\sigma_b^*} - X_{\sigma_b^*}| \leq c_2 \varepsilon) \leq p_1. \quad (3.32)$$

Lemma 3.4 of [3] and its proof show that there exists $c_3 > 0$ such that for all $t \geq 0$, \mathbb{P} -a.s.,

$$|X_t - Y_t| < \exp(c_3(L_t^X + L_t^Y)) |x_0 - y_0|. \quad (3.33)$$

Hence,

$$\inf_{0 \leq t \leq \sigma_b^*} |X_t - Y_t| \geq \exp(-2c_3 b) |X_{\sigma_b^*} - Y_{\sigma_b^*}|. \quad (3.34)$$

Let $c_4 = \exp(-2c_3 b)$ and $c_5 = c_2 c_4$. It follows from (3.32) and (3.34) that

$$\mathbb{P}^{x_0, y_0}\left(\inf_{0 \leq t \leq \sigma_b^*} |Y_t - X_t| \leq c_5 \varepsilon\right) \leq p_1, \quad (3.35)$$

$$\sup_{0 \leq t \leq \sigma_b^*} |Y_t - X_t| \leq c_4 \varepsilon, \quad \text{a.s.} \quad (3.36)$$

We set $c_6 = (-2 - 2c_3 b) \wedge \log c_5$ and $c_7 = e^{c_6}$. Obviously, (3.35) implies that

$$\mathbb{P}^{x_0, y_0}\left(\inf_{0 \leq t \leq \sigma_b^*} |Y_t - X_t| \leq c_7 \varepsilon\right) \leq p_1. \quad (3.37)$$

Let

$$S_1 = \inf\{t \geq 0 : V_t - V_0 \leq c_6\},$$

with convention that $\inf \emptyset = \infty$. Note that at least one of the processes X and Y must belong to ∂D at time S_1 . We will assume that $X_{S_1} \in \partial D$; we will discuss this assumption below. Let

$$U_1 = \inf\{t \geq S_1 : Y_t \in \partial D\}.$$

We proceed by induction. Let

$$S_k = \inf\{t \geq U_{k-1} : V_t - V_{U_{k-1}} \leq c_6\}.$$

Either $X_{S_k} \in \partial D$ or $Y_{S_k} \in \partial D$. We will assume that $X_{S_k} \in \partial D$ in the following definitions. If $Y_{S_k} \in \partial D$ then we exchange the roles of X and Y in the definitions of all objects related to S_k . We will present the argument only in the case $X_{S_k} \in \partial D$ because the estimates hold in the other case by symmetry.

Fix some β_3 and β_4 such that $\beta_1 < \beta_3 < \beta_4 < 1/2$. Let $z_k \in \partial D$ be the point such that $\mathbf{n}(z_k) = \frac{Y_{S_k} - X_{S_k}}{|Y_{S_k} - X_{S_k}|}$, and for some c_7 and c_8 to be specified later,

$$\begin{aligned} F_k &= \{S_k < \sigma_b^*\}, \\ \mathcal{G}_k &= \sigma(B_t, t \leq S_k), \\ J_k &= \min\{n \in \mathbb{Z} : |X_{S_k} - z_k| \geq 2^{-n}\}, \\ U_k &= \inf\{t \geq S_k : Y_t \in \partial D\}, \\ \mathcal{H}_k &= \sigma(B_t, t \leq U_k), \\ d_k &= |X_{U_{k-1}} - Y_{U_{k-1}}|, \\ I_k &= \{2^{-J_k} \geq d_k^{\beta_3}\}, \\ C_k &= \left\{U_k \leq \inf\{t \geq S_k : |X_t - X_{S_k}| \geq d_k^{\beta_4}\}\right\}, \\ G_k &= \{|X_{U_k} - Y_{U_k}| \geq c_7 2^{-J_k} d_k\}, \\ H_k &= \left\{\frac{|\langle X_{U_k} - Y_{U_k}, \mathbf{n}(Y_{U_k}) \rangle|}{|X_{U_k} - Y_{U_k}|} \leq c_8 |X_{U_k} - Y_{U_k}|^{\beta_4}\right\}, \\ A_k &= I_k \cap C_k \cap G_k \cap H_k. \end{aligned}$$

Since $H_{k-1} \subset A_{k-1}$, it follows from (3.37) and the strong Markov property applied at U_{k-1} that on $F_{k-1} \cap \bigcap_{j \leq k-1} A_j$,

$$\mathbb{P}(S_k \leq \sigma_b^* \mid \mathcal{H}_{k-1}) \leq p_1. \quad (3.38)$$

Step 2. In this step we will prove that, for some $c_9, c_{10} < \infty$, $k \geq 1$ and m such that $2^{-m} \geq d_k^{\beta_3}$,

$$\mathbb{P}(A_k^c \mid \mathcal{G}_k) \leq c_9 d_k^{\beta_3} \quad \text{on } F_k \cap \bigcap_{j \leq k-1} A_j, \quad (3.39)$$

$$\mathbb{P}(J_k \geq m \mid \mathcal{G}_k) \leq c_{10} 2^{-m} \quad \text{on } F_k \cap \bigcap_{j \leq k-1} A_j. \quad (3.40)$$

By the definition of S_k , $|X_{S_k} - Y_{S_k}| = c_7 d_k$, if $S_k < \infty$. We have assumed that $X_{S_k} \in \partial D$ so $\text{dist}(Y_{S_k}, \partial D) \leq c_7 d_k$. We apply Lemma 3.2 of [4] to the process Y at the stopping time S_k to obtain

$$\mathbb{P}(C_k^c \mid \mathcal{G}_k) \leq c_{11} d_k^{1-\beta_4} \quad \text{on } F_k \cap \bigcap_{j \leq k-1} A_j. \quad (3.41)$$

Note that the proof of Lemma 3.2 in [4] is presented in the multidimensional setting although that paper is concerned with two-dimensional domains.

We obtain from (3.33),

$$\sup_{U_{k-1} \leq t \leq \sigma_b^*} |X_t - Y_t| \leq c_4 |X_{U_{k-1}} - Y_{U_{k-1}}| = c_4 d_k. \quad (3.42)$$

Let $U_k^* = \sup\{t < U_k : Y_t \in \partial D\}$. It is easy to see that, a.s., $U_k^* < S_k < U_k$, for $k \geq 2$. The case $k = 1$ requires minor modifications so we will omit the proof. Random times U_k^* and U_k are the endpoints of an excursion of Y from ∂D . Suppose that $J_k \geq m$ and $2^{-m} \geq d_k^{\beta_3}$. Then $|X_{S_k} - z_k| \leq 2^{-m+1}$ and, using (3.42),

$$|Y_{S_k} - z_k| \leq |X_{S_k} - z_k| + |X_{S_k} - Y_{S_k}| \leq 2^{-m+1} + c_4 d_k \leq 2^{-m+2}. \quad (3.43)$$

Suppose that $\sup_{U_k^* \leq t \leq S_k, X_t \in \partial D} |Y_t - z_k| \leq c_{12} := 1/400$. We will show that this assumption leads to a contradiction. The assumption and (3.36) imply that, for small ε , $\sup_{U_k^* \leq t \leq S_k, X_t \in \partial D} |X_t - z_k| \leq 2c_{12}$. This in turn implies that for all $t \in [U_k^*, S_k]$ such that $X_t \in \partial D$, the angle between $\mathbf{n}(X_t)$ and $\mathbf{n}(z_k)$ is less than $4c_{12}$. It follows that the angle between $\int_{U_k^*}^{S_k} \mathbf{n}(X_t) dL_t^X$ and $\mathbf{n}(z_k)$ is also smaller than $4c_{12}$. Note that $Y_t \notin \partial D$ for $t \in [U_k^*, S_k]$ by the definition of U_k^* . Thus $\int_{U_k^*}^{S_k} \mathbf{n}(Y_t) dL_t^Y = 0$ and, therefore,

$$X_{S_k} - Y_{S_k} = X_{U_k^*} - Y_{U_k^*} + \int_{U_k^*}^{S_k} \mathbf{n}(X_t) dL_t^X. \quad (3.44)$$

Recall that $X_{S_k} - Y_{S_k}$ is a positive multiple of $-\mathbf{n}(z_k)$, $Y_{U_k^*} \in \partial D$ and $X_{S_k} \in \partial D$. This and the fact that the angle between $\int_{U_k^*}^{S_k} \mathbf{n}(X_t) dL_t^X$ and $\mathbf{n}(z_k)$ is smaller than $4c_{12}$ show that (3.44) cannot be true. This contradiction implies that $\sup_{U_k^* \leq t \leq S_k, X_t \in \partial D} |Y_t - z_k| \geq c_{12}$. We combine this with (3.43) to see that $\sup_{U_k^* \leq t \leq S_k, X_t \in \partial D} |Y_t - Y_{S_k}| \geq c_{13} := c_{12}/2$, for some m_1 and all $m \geq m_1$. Suppose that s_1 is such that $U_k^* \leq s_1 \leq S_k$, $X_{s_1} \in \partial D$ and $|Y_{s_1} - Y_{S_k}| \geq c_{13}$. Then either $|Y_{U_k^*} - Y_{S_k}| \geq c_{13}/2$ or $|Y_{U_k^*} - Y_{s_1}| \geq c_{13}/2$. Since $X_{S_k} \in \partial D$, it follows that there

exists s_2 such that $U_k^* \leq s_2 \leq S_k$, $X_{s_2} \in \partial D$ and $|Y_{U_k^*} - Y_{s_2}| \geq c_{13}/2$. We record this for future reference,

$$\sup_{U_k^* \leq t \leq S_k, X_t \in \partial D} |Y_t - Y_{U_k^*}| \geq c_{13}/2. \quad (3.45)$$

Recall notation related to excursions from Section 2.3. We will determine the H^x -measure of the event that an excursion closely approaches ∂D away from its endpoints. More precisely, consider an arbitrary $c_{14} \in (0, c_{13}/2)$ and let

$$\begin{aligned} T &= \inf\{t \geq 0 : \text{dist}(e(t), \partial D) \leq c_4 d_k, |e(0) - e(t)| \geq c_{14}\}, \\ \widehat{A} &= \{T < \zeta, |e(\zeta-) - e(T)| \geq c_{14}\}, \\ \widetilde{A} &= \left\{T < \zeta, \sup_{T < t < \zeta} |e(\zeta-) - e(t)| \geq d_k^{\beta_4}\right\}. \end{aligned}$$

An application of Lemma 3.2 of [4] and (2.2) give

$$H^x(T < \zeta) \leq H^x\left(\sup_{0 < t < \zeta} |e(t) - e(0)| \geq c_{14}\right) \leq c_{15}. \quad (3.46)$$

Another application of Lemma 3.2 of [4] and the strong Markov property at the stopping time T yield

$$H^x(\widehat{A} \mid T < \zeta) = H^x(|e(\zeta-) - e(T)| \geq c_{14} \mid T < \zeta) \leq c_{16} c_4 d_k. \quad (3.47)$$

We combine this and (3.46) to see that,

$$H^x(\widehat{A}) \leq c_{17} d_k. \quad (3.48)$$

The exit system formula (2.1) implies that the probability that there exists an excursion of Y belonging to the set \widehat{A} and starting in the time interval $[U_{k-1}, \sigma_b^*]$ is less than $bc_{17}d_k$. Let

$$I_k^1 = \{\exists t \in [U_k^*, U_k] : X_t \in \partial D, |X_t - X_{U_k^*}| \wedge |X_t - X_{U_k}| \geq c_{14}/2\}.$$

It follows from (3.36) that if $X_t \in \partial D$ for some $0 \leq t \leq \sigma_b^*$ then $\text{dist}(Y_t, \partial D) \leq c_4 \varepsilon$. Assume that ε is so small that $c_4 \varepsilon < c_{14}/2$. If I_k^1 occurred then Y had an excursion belonging to the set \widehat{A} . We have proved that the probability of this event is bounded by $c_{17}bd_k$. Thus,

$$\mathbb{P}(\{J_k \geq m\} \cap I_k^1 \mid \mathcal{G}_k) \leq c_{17}bd_k \quad \text{on } F_k \cap \bigcap_{j \leq k-1} A_j. \quad (3.49)$$

The following is a special case of (3.49), with m defined by $2^{-m-1} \leq d_k^{\beta_3} < 2^{-m}$,

$$\mathbb{P}(I_k^c \cap I_k^1 \mid \mathcal{G}_k) \leq c_{17}bd_k \quad \text{on } F_k \cap \bigcap_{j \leq k-1} A_j. \quad (3.50)$$

Let

$$\begin{aligned} S_k^1 &= \inf\{t \geq U_k^* : X_t \in \partial D, |Y_t - Y_{U_k^*}| \geq c_{14}\}, \\ C_k^1 &= \left\{U_k \leq \inf\{t \geq S_k^1 : |X_t - X_{S_k^1}| \geq d_k^{\beta_4}\}\right\}. \end{aligned}$$

The same type of argument which was used to show (3.47) and (3.48) gives

$$H^x(\tilde{A} \mid T < \zeta) = H^x\left(\sup_{T < t < \zeta} |e(\zeta-) - e(t)| \geq d_k^{\beta_4} \mid T < \zeta\right) \leq c_{18}d_k^{1-\beta_4},$$

and

$$H^x(\tilde{A}) \leq c_{19}d_k^{1-\beta_4}.$$

The exit system formula (2.1) implies that the probability that there exists an excursion of Y belonging to the set \tilde{A} and starting in the time interval $[U_{k-1}, \sigma_b^*]$ is less than $bc_{19}d_k^{1-\beta_4}$. It follows from (3.36) that if $X_t \in \partial D$ for some $U_{k-1} \leq t \leq \sigma_b^*$ then $\text{dist}(Y_t, \partial D) \leq c_4d_k$. For small $\varepsilon > 0$, d_k is also small so $c_4d_k < d_k^{\beta_4}$. Therefore if C_k^1 occurred then Y had an excursion belonging to the set \tilde{A} . We have proved that the probability of this event is bounded by $c_{19}bd_k^{1-\beta_4}$. Thus,

$$\mathbb{P}((C_k^1)^c \mid \mathcal{G}_k) \leq c_{19}d_k^{1-\beta_4} \quad \text{on } F_k \cap \bigcap_{j \leq k-1} A_j. \quad (3.51)$$

Let $c_{20} > 0$ be such that there are no vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ satisfying these conditions: (i) $|\mathbf{w}| \geq c_7|\mathbf{v}| > 0$, (ii) the angle between \mathbf{v} and \mathbf{w} is greater than α , and (iii) the angle between $\mathbf{w} - \mathbf{v}$ and \mathbf{w} is less than α/c_{20} .

Suppose that the event $\{J_k \in [m, m+1]\} \cap (I_k^1)^c \cap C_k \cap C_k^1$ occurred for some m such that $2^{-m} \geq d_k^{\beta_3}$. Note that $S_k^1 \leq S_k$ because of (3.45). Let α_k be the angle between $X_{S_k^1} - Y_{S_k^1}$ and $X_{S_k} - Y_{S_k}$. We will consider two cases, when $\alpha_k \geq 4c_{20}c_{22}2^{-J_k}$ and $\alpha_k \leq 4c_{20}c_{22}2^{-J_k}$, where c_{22} is a constant whose value is specified below.

Suppose that $\alpha_k \geq 4c_{20}c_{22}2^{-J_k}$. For all $t \in [S_k^1, S_k]$ such that $X_t \in \partial D$, the angle between $\mathbf{n}(X_t)$ and $\mathbf{n}(X_{S_k})$ is smaller than $c_{21}d_k^{\beta_4}$ because C_k^1 holds. It follows that the angle between $\int_{S_k^1}^{S_k} \mathbf{n}(X_t)dL_t^X$ and $\mathbf{n}(X_{S_k})$ is also smaller than $c_{21}d_k^{\beta_4}$. Since $J_k \geq m$, the angle between $\mathbf{n}(z_1)$ and $\mathbf{n}(X_{S_k})$ is smaller than or equal to 2^{-m} . This is equivalent to saying that the angle between $X_{S_k} - Y_{S_k}$ and $\mathbf{n}(X_{S_k})$ is smaller than or equal to 2^{-m} . It follows that the angle between $\int_{S_k^1}^{S_k} \mathbf{n}(X_t)dL_t^X$ and $X_{S_k} - Y_{S_k}$ is smaller than $2^{-m} + c_{21}d_k^{\beta_4} \leq 2^{-m} + c_{21}d_k^{\beta_3} \leq 2^{-m} + c_{21}2^{-m} = c_{22}2^{-m}$ (this defines c_{22}). Note that $Y_t \notin \partial D$ for $t \in [S_k^1, S_k]$. Thus $\int_{S_k^1}^{S_k} \mathbf{n}(Y_t)dL_t^Y = 0$ and, therefore,

$$X_{S_k} - Y_{S_k} = X_{S_k^1} - Y_{S_k^1} + \int_{S_k^1}^{S_k} \mathbf{n}(X_t)dL_t^X. \quad (3.52)$$

It follows from the definition of S_k and the fact that $S_k^1 \in [U_{k-1}, S_k]$ that $|X_{S_k} - Y_{S_k}| \geq c_7|X_{S_k^1} - Y_{S_k^1}|$. Recall that α_k , the angle between $X_{S_k^1} - Y_{S_k^1}$ and $X_{S_k} - Y_{S_k}$, is assumed to be greater than $4c_{20}c_{22}2^{-J_k} \geq 4c_{20}c_{22}2^{-m-1} = 2c_{20}c_{22}2^{-m}$. This, the fact that the angle between $\int_{S_k^1}^{S_k} \mathbf{n}(X_t)dL_t^X$ and $X_{S_k} - Y_{S_k}$ is smaller than $c_{22}2^{-m}$, the definition of c_{20} and (3.52) yield a

contradiction. Hence, we must have $\alpha_k \leq 4c_{20}c_{21}2^{-J_k}$ if $\{J_k \in [m, m+1]\} \cap (I_k^1)^c \cap C_k \cap C_k^1$ holds.

If C_k^1 occurred then $\sup_{S_k^1 \leq t \leq U_k} |X_t - X_{S_k^1}| \leq d_k^{\beta_4}$. Since $S_k \in [S_k^1, U_k]$, it follows that $|X_{U_k} - X_{S_k}| \leq 2d_k^{\beta_4}$. This implies that

$$|Y_{U_k} - X_{S_k}| \leq |X_{U_k} - X_{S_k}| + |X_{U_k} - Y_{U_k}| \leq 2d_k^{\beta_4} + c_4 d_k \leq c_{23} d_k^{\beta_4}.$$

Let $z_k^1 \in \partial D$ be defined by $\mathbf{n}(z_k^1) = (Y_{S_k^1} - X_{S_k^1})/|Y_{S_k^1} - X_{S_k^1}|$. If we assume that $\alpha_k \leq 4c_{20}c_{22}2^{-J_k}$ then $|z_k - z_k^1| \leq c_{24}2^{-J_k} \leq c_{24}2^{-m}$. If $2^{-m} \geq d_k^{\beta_3}$ then

$$\begin{aligned} |Y_{U_k} - z_k^1| &\leq |Y_{U_k} - X_{S_k}| + |X_{S_k} - z_k| + |z_k - z_k^1| \\ &\leq c_{23}d_k^{\beta_4} + 2^{-J_k+1} + c_{24}2^{-m} \leq c_{23}d_k^{\beta_4} + 2^{-m+1} + c_{24}2^{-m} \leq c_{25}2^{-m}. \end{aligned} \quad (3.53)$$

The event in (3.53) expresses a joint property of an excursion of Y from ∂D over the interval $[U_k^*, U_k]$ and the process X , because the definition of z_k^1 involves X . We will estimate the probability of this event using excursion theory.

Let T_j^1 be the starting time of the j -th excursion e_j of Y from ∂D with the property that

$$\sup_{0 \leq t < \zeta} |e_j(0) - e_j(t)| \geq c_{14}, \quad (3.54)$$

and let

$$\begin{aligned} T_j^2 &= \inf\{t \geq T_j^1 : |Y_{T_j^1} - Y_t| \geq c_{14}\}, \\ T_j^3 &= \inf\{t \geq T_j^2 : X_t \in \partial D\}, \\ T_j^4 &= \inf\{t \geq T_j^2 : Y_t \in \partial D\}. \end{aligned}$$

The number of excursions e_j starting before σ_b^* and such that $T_j^2 \leq T_j^4$ is Poisson with the mean bounded by $c_{26}b$, by (2.1). Consider an excursions e_j such that $T_j^3 \leq T_j^4$. Note that T_j^2 is a stopping time for Y (although T_j^1 's are not stopping times). Let $z_j^2 \in \partial D$ be the point such that $\mathbf{n}(z_j^2) = (Y_{T_j^2} - X_{T_j^2})/|Y_{T_j^2} - X_{T_j^2}|$. Since neither X nor Y visit ∂D during the interval $[T_j^2, T_j^3]$, we have $\mathbf{n}(z_j^2) = (Y_{T_j^3} - X_{T_j^3})/|Y_{T_j^3} - X_{T_j^3}|$. We can assume that $c_{14} > 0$ is arbitrarily small. If c_{14} is sufficiently small then it is easy to see that the angle between $Y_{T_j^2} - X_{T_j^2}$ and $\mathbf{n}(Y_{T_j^1})$ must be bounded below by a strictly positive constant and, therefore, the distance between z_j^2 and $Y_{T_j^2}$ must be bounded below by $c_{27} > 0$. Note that z_j^2 is measurable with this respect to the σ -field $\mathcal{F}_{T_j^2}$, so we can apply the strong Markov property at T_j^2 to obtain the following estimate. Given the values of $Y_{T_j^2}$ and z_j^2 , and assuming that $|Y_{T_j^2} - z_j^2| \geq c_{27}$, the probability that $Y_{T_j^4} \in \mathcal{B}(z_j^2, c_{25}2^{-m})$ is smaller than $c_{28}2^{-m}$, by the standard estimates for the hitting distribution of the sphere. Hence the expected number of excursions e_j starting before σ_b^* , such that (3.54) holds and $Y_{T_j^4} \in \mathcal{B}(z_j^2, c_{25}2^{-m})$ is bounded by $c_{28}b2^{-m}$. This implies that the probability that such an excursion will occur is less than

or equal to $c_{28}b2^{-m}$. If the event in (3.53) occurred then there exists an excursion e_j with the properties described above. Taken together, these observations prove that,

$$\mathbb{P}(\{J_k \in [m, m+1]\} \cap (I_k^1)^c \cap C_k \cap C_k^1 \mid \mathcal{G}_k) \leq c_{28}2^{-m} \quad \text{on } F_k \cap \bigcap_{j \leq k-1} A_j.$$

Summing over $m \geq m'$, we obtain

$$\mathbb{P}(\{J_k \geq m'\} \cap (I_k^1)^c \cap C_k \cap C_k^1 \mid \mathcal{G}_k) \leq c_{29}2^{-m'} \quad \text{on } F_k \cap \bigcap_{j \leq k-1} A_j. \quad (3.55)$$

We combine (3.41), (3.49), (3.51) and (3.55) to see that (3.40) holds.

The following is a special case of (3.55), with m' defined by $2^{-m'-1} \leq d_k^{\beta_3} < 2^{-m'}$,

$$\mathbb{P}(I_k^c \cap (I_k^1)^c \cap C_k \cap C_k^1 \mid \mathcal{G}_k) \leq c_{30}d_k^{\beta_3} \quad \text{on } F_k \cap \bigcap_{j \leq k-1} A_j. \quad (3.56)$$

Let

$$\begin{aligned} \widehat{S}_k^j &= \inf\{t \geq S_k : |X_t - Y_t| \leq 2^{-j}\}, \\ \widehat{C}_k^j &= \left\{U_k \leq \inf\{t \geq \widehat{S}_k^j : |X_t - X_{\widehat{S}_k^j}| \geq 2^{-j\beta_4}\}\right\}. \end{aligned}$$

If $\widehat{S}_k^j \leq U_k$ then $X_{\widehat{S}_k^j} \in \partial D$. Thus the following estimate can be proved just like (3.41),

$$\mathbb{P}((\widehat{C}_k^j)^c \cap \{\widehat{S}_k^j \leq U_k\} \mid \mathcal{G}_k) \leq c_{31}2^{-j(1-\beta_4)} \quad \text{on } F_k \cap \bigcap_{j \leq k-1} A_j. \quad (3.57)$$

Let j_0 be the largest j such that \widehat{S}_k^j holds. Since $X_{\widehat{S}_k^{j_0}} \in \partial D$ and $Y_{U_k} \in \partial D$, there is $t \in [\widehat{S}_k^{j_0}, U_k]$ such that $\text{dist}(X_t, \partial D) = \text{dist}(Y_t, \partial D)$. Let $\widetilde{S}_k^{j_0}$ be the smallest $t \geq \widehat{S}_k^{j_0}$ with this property. We apply Lemma 3.1 at the stopping time $\widetilde{S}_k^{j_0}$ to see that, if c_8 in the definition of H_k is chosen appropriately then $\widehat{C}_k^{j_0} \subset H_k$. This and (3.57) imply that

$$\mathbb{P}(H_k^c \cap C_k \mid \mathcal{G}_k) \leq \mathbb{P}\left(\bigcup_{j: 2^{-j-1} \leq d_k^{\beta_4}} (\widehat{C}_k^j)^c \cap \{\widehat{S}_k^j \leq U_k\} \mid \mathcal{G}_k\right) \leq c_{32}d_k^{1-\beta_4} \quad \text{on } F_k \cap \bigcap_{j \leq k-1} A_j. \quad (3.58)$$

Suppose that $I_k \cap C_k \cap H_k$ holds. Recall that $\beta_4 > \beta_3$. This implies that for small ε_1 , $d_k^{\beta_4}$ is much smaller than $d_k^{\beta_3}$ and, therefore, $d_k^{\beta_4}$ is much smaller than 2^{-J_k} . For all $t \in [S_k, U_k]$ such that $X_t \in \partial D$, the angle between $\mathbf{n}(X_t)$ and $\mathbf{n}(X_{S_k})$ is smaller than $c_{33}d_k^{\beta_4}$. It follows that the angle between $\int_{S_k}^{U_k} \mathbf{n}(X_t)dL_t^X$ and $\mathbf{n}(X_{S_k})$ is also smaller than $c_{33}d_k^{\beta_4}$. The angle between $\mathbf{n}(X_{S_k})$ and $\mathbf{n}(z_k)$ is greater than 2^{-J_k} . This implies that the angle between $\int_{S_k}^{U_k} \mathbf{n}(X_t)dL_t^X$ and $\mathbf{n}(z_k)$ is greater than 2^{-J_k-1} . Note that $Y_t \notin \partial D$ for $t \in [S_k, U_k]$ by the definition of U_k . Thus $\int_{S_k}^{U_k} \mathbf{n}(Y_t)dL_t^Y = 0$ and, therefore,

$$X_{U_k} - Y_{U_k} = X_{S_k} - Y_{S_k} + \int_{S_k}^{U_k} \mathbf{n}(X_t)dL_t^X. \quad (3.59)$$

Recall that $X_{S_k} - Y_{S_k}$ is parallel to $\mathbf{n}(z_k)$, by definition. This, the fact that the angle between $\int_{S_k}^{U_k} \mathbf{n}(X_t) dL_t^X$ and $\mathbf{n}(z_k)$ is greater than 2^{-J_k-1} , and (3.59) easily imply that $|X_{U_k} - Y_{U_k}| \geq c_{34}2^{-J_k}|X_{S_k} - Y_{S_k}| = c_{34}2^{-J_k}d_k/2$. We see that if we take $c_7 = c_{34}/4$ in the definition of G_k then

$$\mathbb{P}(G_k^c \cap I_k \cap C_k \cap H_k \mid \mathcal{G}_k) = 0 \quad \text{on } F_k \cap \bigcap_{j \leq k-1} A_j. \quad (3.60)$$

It follows from (3.41), (3.50), (3.51), (3.56), (3.58) and (3.60) that on $F_k \cap \bigcap_{j \leq k-1} A_j$,

$$\begin{aligned} \mathbb{P}(A_k^c \mid \mathcal{G}_k) &= \mathbb{P}(I_k^c \cup C_k^c \cup G_k^c \cup H_k^c \mid \mathcal{G}_k) \\ &\leq \mathbb{P}(C_k^c \mid \mathcal{G}_k) + \mathbb{P}(I_k^c \cap I_k^1 \mid \mathcal{G}_k) + \mathbb{P}((C_k^1)^c \mid \mathcal{G}_k) + \mathbb{P}(I_k^c \cap (I_k^1)^c \cap C_k \cap C_k^1 \mid \mathcal{G}_k) \\ &\quad + \mathbb{P}(C_k \cap H_k^c \mid \mathcal{G}_k) + \mathbb{P}(G_k^c \cap I_k \cap C_k \cap H_k \mid \mathcal{G}_k) \\ &\leq c_{35}d_k^{\beta_3}. \end{aligned}$$

This completes the proof of (3.39).

Step 3. We return to the main argument. It follows from (3.33) and the definition of c_6 that if F_k occurred then

$$\sup_{S_k \leq t \leq \sigma_b^*} |Y_t - X_t| \leq d_k/2.$$

Hence, if the event $\bigcap_{j \leq k} F_j$ occurred then $\sup_{S_k \leq t \leq \sigma_b^*} |Y_t - X_t| \leq \varepsilon 2^{-k}$, and, therefore,

$$|Y_{U_k} - X_{U_k}| \leq \varepsilon 2^{-k}.$$

This and (3.39) imply that

$$\mathbb{P}\left(A_k^c \cap F_k \cap \bigcap_{j \leq k-1} A_j\right) \leq c_9 d_k^{\beta_3} \leq c_9 \varepsilon^{\beta_3} 2^{-(k-1)\beta_3}. \quad (3.61)$$

Let

$$F = F_1^c \cup \bigcup_{k=1}^{\infty} \left(F_k \cap F_{k+1}^c \cap \bigcap_{j \leq k} A_j \right).$$

If $\bigcap_{k=1}^{\infty} F_k$ holds then $\inf_{0 \leq t \leq \sigma_b^*} |X_t - Y_t| = 0$. The last event has probability 0, according to Lemma 3.2 so $\mathbb{P}(\bigcap_{k=1}^{\infty} F_k) = 0$. Since $F_{k+1} \subset F_k$, there exists at most one N_1 such that $F_{N_1}^c \cup F_{N_1+1}$ fails. There exists at most one N_2 be such that A_j holds for all $j < N_2$ and A_{N_2}

does not hold. Thus, by (3.61),

$$\begin{aligned}
\mathbb{P}(F^c) &= \mathbb{P}\left(F_1 \cap \bigcap_{k=1}^{\infty} \left(F_k^c \cup F_{k+1} \cup \bigcup_{j \leq k} A_j^c\right)\right) \\
&\leq \mathbb{P}(F_1 \cap \{N_1 = \infty\}) + \mathbb{P}\left(\bigcap_{k=1}^{\infty} \left(F_k^c \cup F_{k+1} \cup \bigcup_{j \leq k} A_j^c\right) \cap \{1 \leq N_1 < \infty\}\right) \\
&= \mathbb{P}\left(\bigcap_{k=1}^{\infty} F_k \cap \{N_1 = \infty\}\right) + \mathbb{P}\left(\bigcap_{k=1}^{\infty} \left(F_k^c \cup F_{k+1} \cup \bigcup_{j \leq k} A_j^c\right) \cap \{N_1 < \infty\}\right) \\
&\leq 0 + \mathbb{P}\left(\bigcup_{n=1}^{\infty} \left(\left(F_n^c \cup F_{n+1} \cup \bigcup_{j \leq n} A_j^c\right) \cap \{N_1 = n\}\right)\right) \\
&= \mathbb{P}\left(\bigcup_{n=1}^{\infty} \left(\left(\bigcup_{j \leq n} A_j^c\right) \cap \{N_1 = n\}\right)\right) \\
&= \mathbb{P}\left(\bigcup_{n=1}^{\infty} \bigcup_{m=1}^n \left(\left(\bigcup_{j \leq m} A_j^c\right) \cap \{N_1 = n, N_2 = m\}\right)\right) \\
&= \mathbb{P}\left(\bigcup_{m=1}^{\infty} \left(\left(\bigcup_{j \leq m} A_j^c\right) \cap \{N_1 \geq m, N_2 = m\}\right)\right) \\
&\leq \mathbb{P}\left(\bigcup_{m=1}^{\infty} \left(\bigcap_{j \leq m} A_j \cap A_m^c \cap F_m\right)\right) \\
&\leq \sum_{m=1}^{\infty} \mathbb{P}\left(\bigcap_{j \leq m} A_j \cap A_m^c \cap F_m\right) \\
&\leq \sum_{m=1}^{\infty} c_9 \varepsilon^{\beta_3} 2^{-(m-1)\beta_3} \leq c_{36} \varepsilon^{\beta_3}.
\end{aligned}$$

This proves (3.30).

Since $F_{k+1} \subset F_k$ and $(F_{k+1} \cap \bigcap_{j \leq k+1} A_j) \subset (F_k \cap \bigcap_{j \leq k} A_j)$, we have

$$\begin{aligned}
F &= F_1^c \cup \bigcup_{n=1}^{\infty} \left(F_n \cap F_{n+1}^c \cap \bigcap_{j \leq n} A_j \right) \\
&= F_1^c \cup \bigcup_{n=1}^{k-2} \left(F_n \cap F_{n+1}^c \cap \bigcap_{j \leq n} A_j \right) \cup \left(F_{k-1} \cap F_k^c \cap \bigcap_{j \leq k-1} A_j \right) \\
&\quad \cup \bigcup_{n=k}^{\infty} \left(F_n \cap F_{n+1}^c \cap \bigcap_{j \leq n} A_j \right) \\
&\subset F_1^c \cup \bigcup_{n=1}^{k-2} F_{n+1}^c \cup (F_{k-1} \cap F_k^c) \cup \bigcup_{n=k}^{\infty} \left(F_n \cap \bigcap_{j \leq n} A_j \right) \\
&\subset F_{k-1}^c \cup (F_{k-1} \cap F_k^c) \cup \left(F_k \cap \bigcap_{j \leq k} A_j \right). \tag{3.62}
\end{aligned}$$

Let $T_k = U_k \wedge \sigma_b^*$. We claim that

$$\left| \log |X_{T_k} - Y_{T_k}| - \log |X_{T_{k-1}} - Y_{T_{k-1}}| \right| \begin{cases} = 0, & \text{if } F_{k-1}^c \text{ holds;} \\ \leq c_6, & \text{if } F_{k-1} \cap F_k^c \text{ holds;} \\ \leq c_{37}m & \text{if } J_k = m \text{ and } F_k \cap \bigcap_{j \leq k} A_j \text{ holds.} \end{cases} \tag{3.63}$$

The first claim follows from the definitions of T_k , S_k and F_{k-1} . The second claim follows from the definition of S_k . The last claim follows from the fact that $G_k \subset A_k$.

If $F_k \cap A_k$ holds then H_k holds and we can apply (3.37) at the stopping time U_k . By the repeated application of the strong Markov property we obtain $\mathbb{P}(F_{k-1} \mid \mathcal{H}_{k-1}) \leq p_1^{k-1}$. It follows from (3.38), (3.40) and (3.63) that

$$\mathbb{P}(\{J_k \geq m\} \cap F_k \cap A_k \mid \mathcal{H}_{k-1}) \leq c_{38}2^{-m} \quad \text{on } F_{k-1} \cap \bigcap_{j \leq k-1} A_j, \tag{3.64}$$

$$\mathbb{P}(|\log |X_{T_k} - Y_{T_k}| - \log |X_{T_{k-1}} - Y_{T_{k-1}}|| \geq c_6 \mid \mathcal{H}_{k-1}) \leq 1 - p_1^{k-1}, \tag{3.64}$$

$$\mathbb{P}(|\log |X_{T_k} - Y_{T_k}| - \log |X_{T_{k-1}} - Y_{T_{k-1}}|| \geq c_{37}m \mid \mathcal{H}_{k-1}) \leq 1 - p_1^{k-1}c_{38}2^{-m}. \tag{3.65}$$

Let k_0 be such that $p_1^{k-1} + p_1^{k-1} \sum_{m \geq 1} c_{38}2^{-m} \leq 1$ for $k \geq k_0$ and let m_0 be such that $\sum_{m \geq m_0} c_{38}2^{-m} \leq 1$. Let $q' \geq 0$ be such that $q' + \sum_{m \geq m_0} c_{38}2^{-m} = 1$ and let $q_k \geq 0$ be such that $q_k + p_1^{k-1} + p_1^{k-1} \sum_{m \geq 1} c_{38}2^{-m} = 1$, for $k \geq k_0$. Let Z_k , $k \geq 1$, be independent random variables with the following distributions; for $1 \leq k \leq k_0 - 1$,

$$Z_k = \begin{cases} c_6 + c_{37}m_0, & \text{with probability } q'; \\ c_{37}m & \text{with probability } c_{38}2^{-m} \text{ for } m \geq m_0, \end{cases}$$

and for $k \geq k_0$,

$$Z_k = \begin{cases} 0, & \text{with probability } q_k; \\ c_6, & \text{with probability } p_1^{k-1}; \\ c_{37}m & \text{with probability } c_{38}p_1^{k-1}2^{-m} \text{ for } m \geq 1. \end{cases}$$

By (3.62), (3.64) and (3.65), the random variable

$$|V_1 - V_0|\mathbf{1}_F = \left| \sum_{k=1}^{\infty} \log |X_{T_k} - Y_{T_k}| - \log |X_{T_{k-1}} - Y_{T_{k-1}}| \right| \mathbf{1}_F$$

is stochastically dominated by $Z_* := \sum_{k \geq 1} Z_k$. To finish the proof of (3.31), it remains to show that $\mathbb{E}Z_* < \infty$. Indeed, we have

$$\begin{aligned} \mathbb{E}Z_* &= \sum_{1 \leq k \leq k_0-1} \left((c_6 + c_{37}m_0)q' + \sum_{m \geq m_0} c_{38}2^{-m}c_{37}m \right) \\ &\quad + \sum_{k \geq k_0} \left(q \cdot 0 + p_1^{k-1}c_6 + p_1^{k-1} \sum_{m \geq 1} c_{38}2^{-m}c_{37}m \right) < \infty. \end{aligned}$$

□

Recall notation from (3.6).

Lemma 3.6. *For any $\beta_1 \in (0, 1/2)$ there exist $\beta_2, c_1, b, \varepsilon_1 > 0$ such that if $\varepsilon \leq \varepsilon_1$, $x_0 \in \partial D$, $y_0 \in \overline{D}$, $|x_0 - y_0| = \varepsilon$, $X_0 = x_0$, $Y_0 = y_0$ and*

$$\frac{|\langle y_0 - x_0, \mathbf{n}(x_0) \rangle|}{|y_0 - x_0|} \leq \varepsilon^{\beta_1} \quad (3.66)$$

then there exists an event F such that

$$\mathbb{P}^{x_0, y_0}(F^c) \leq \varepsilon^{\beta_2}, \quad (3.67)$$

$$\mathbb{E}^{x_0, y_0}((V_1 - V_0)\mathbf{1}_F) \geq c_1. \quad (3.68)$$

Proof. We will use the following elementary fact, the proof of which is left to the reader. Suppose that a cumulative distribution function $G : \mathbb{R} \rightarrow [0, 1]$ satisfies $\int_{-\infty}^{\infty} |a| dG(a) < \infty$. Then for every $c_2 > 0$ there exist $p_1, p_2 > 0$ such that if Z is a random variable and F is an event satisfying $\mathbb{P}(F^c) \leq p_1$, $\mathbb{P}(Z\mathbf{1}_F \leq a) \leq G(a)$ for $a \in \mathbb{R}$ and $\mathbb{P}(Z\mathbf{1}_F \leq c_2) \leq p_2$ then $\mathbb{E}Z \geq c_2/2$. We will apply this observation to $Z = V_1 - V_0$. By Lemma 3.5, there exists an event F such that $\mathbb{P}(F^c) \leq \varepsilon^{\beta_2}$ and $\mathbb{P}((V_1 - V_0)\mathbf{1}_F \leq a) \leq G(a)$ for $a \in \mathbb{R}$ for some G with $\int_{-\infty}^{\infty} |a| dG(a) < \infty$. The condition $\mathbb{P}((V_1 - V_0)\mathbf{1}_F \leq c_2) \leq p_2$ is satisfied by Lemma 3.4. It follows that $\mathbb{E}^{x_0, y_0}((V_1 - V_0)\mathbf{1}_F) \geq c_2/2$. This proves the lemma.

□

Proof of Theorem 1.1. Recall that $\sigma_b^* = \sigma_b^X \wedge \sigma_b^Y$ and

$$\sigma_{(k+1)b}^* = \inf \left\{ t \geq \sigma_{kb}^* : \left(L_t^X - L_{\sigma_{kb}^*}^X \right) \wedge \left(L_t^Y - L_{\sigma_{kb}^*}^Y \right) \geq b \right\},$$

for $k \geq 2$. Fix some $\varepsilon_1, b, \beta_1 > 0$; we will choose values for these parameters later in the proof.

We will define processes X_t^* and Y_t^* for $t \geq 0$ in an inductive way. Let $X_t^* = X_t$ and $Y_t^* = Y_t$ for $t \in [0, \sigma_b^*)$.

By Lemma 3.2, $Y_{\sigma_b^*} \neq X_{\sigma_b^*}$, $\mathbb{P}^{x,y}$ -a.s., for any $x, y \in \overline{D}$, $x \neq y$. Fix an arbitrary $p_1 > 0$ and choose $c_1 > 0$ such that

$$\mathbb{P}^{x,y}(|Y_{\sigma_b^*} - X_{\sigma_b^*}| \leq c_1) < p_1. \quad (3.69)$$

Let $F_1 = \{|Y_{\sigma_b^*} - X_{\sigma_b^*}| \geq c_1\}$.

Suppose that $\sigma_b^* = \sigma_b^X$. Fix some c_2 and let

$$A_1 = \left\{ \frac{|\langle Y_{\sigma_b^*} - X_{\sigma_b^*}, \mathbf{n}(X_{\sigma_b^*}) \rangle|}{|Y_{\sigma_b^*} - X_{\sigma_b^*}|} \leq c_2 |Y_0 - X_0|^{\beta_1} \right\},$$

$$\begin{cases} Y_{\sigma_b^*}^* = Y_{\sigma_b^*} & \text{if } A_1 \cap F_1 \text{ holds,} \\ Y_{\sigma_b^*}^* = X_{\sigma_b^*} + \pi_{X_{\sigma_b^*}}(Y_{\sigma_b^*} - X_{\sigma_b^*}) \frac{|X_{\sigma_b^*} - Y_{\sigma_b^*}| \vee c_1}{|\pi_{X_{\sigma_b^*}}(Y_{\sigma_b^*} - X_{\sigma_b^*})|} & \text{otherwise.} \end{cases} \quad (3.70)$$

Let $\{Y_t^*, t \in [\sigma_b^*, \sigma_{2b}^*)\}$ be the solution to (1.2) with the initial condition given by (3.70) and driven by Brownian motion $\{B_t, t \in [\sigma_b^*, \sigma_{2b}^*)\}$. Let $X_t^* = X_t$ for $t \in [\sigma_b^*, \sigma_{2b}^*)$. Whether A_1 holds or not, we have $|Y_{\sigma_b^*}^* - X_{\sigma_b^*}^*| \geq |Y_{\sigma_b^*} - X_{\sigma_b^*}|$ and

$$\frac{|\langle Y_{\sigma_b^*}^* - X_{\sigma_b^*}^*, \mathbf{n}(X_{\sigma_b^*}^*) \rangle|}{|Y_{\sigma_b^*}^* - X_{\sigma_b^*}^*|} \leq c_2 |Y_0 - X_0|^{\beta_1}. \quad (3.71)$$

Note that

$$\mathbb{E}^{x,y} \log |Y_{\sigma_b^*}^* - X_{\sigma_b^*}^*| \geq \log c_1 > -\infty. \quad (3.72)$$

If $\sigma_b^* = \sigma_b^Y$ then we exchange the roles of X and Y in the above definitions.

We will define some events and processes using induction. The meaning of the following notation will be fully explained below. Let

$$\begin{aligned} \sigma_0^{**} &= 0, \\ \sigma_{kb}^{**} &= \inf \left\{ t \geq \sigma_{(k-1)b}^{**} : \left(L_t^{X^*} - L_{\sigma_{(k-1)b}^{**}}^{X^*} \right) \wedge \left(L_t^{Y^*} - L_{\sigma_{(k-1)b}^{**}}^{Y^*} \right) \geq b \right\}, \quad k \geq 1, \\ R_t^* &= |X_t^* - Y_t^*|, \quad M_t^* = \log R_t^*, \quad t \geq 0, \\ V_k^* &= M_{\sigma_{kb}^{**}}^*, \quad k = 0, 1, \dots \end{aligned}$$

In view of (3.71), we can apply Lemma 3.6 to the process $\{(X_t^*, Y_t^*), t \in [\sigma_b^{**}, \sigma_{2b}^{**})\}$ to conclude that if $\varepsilon_1 > 0$ is sufficiently small then there exist $c_3 > 0$ and an event $F_2 \in$

$\sigma((X_t^*, Y_t^*), t \in [\sigma_b^{**}, \sigma_{2b}^{**}))$ such that, a.s. on the event $\{R_{\sigma_b^{**}}^* \leq \varepsilon_1\}$,

$$\begin{aligned}\mathbb{P}\left(F_2^c \mid X_{\sigma_b^{**}}^*, Y_{\sigma_b^{**}}^*\right) &\leq (R_{\sigma_b^{**}}^*)^{\beta_3}, \\ \mathbb{E}\left((V_2^* - V_1^*) \mathbf{1}_{F_2} \mid X_{\sigma_b^{**}}^*, Y_{\sigma_b^{**}}^*\right) &\geq c_3.\end{aligned}$$

We proceed with the inductive definition. Suppose that F_k , X_t^* and Y_t^* are already defined for some $k \geq 2$ and $t \in [0, \sigma_{kb}^{**})$. Suppose that $\sigma_{kb}^{**} = \inf \left\{ t \geq \sigma_{(k-1)b}^{**} : L_t^{X^*} - L_{\sigma_{(k-1)b}^{**}}^{X^*} \geq b \right\}$ and let

$$\begin{aligned}A_k &= \left\{ \frac{\left| \left\langle Y_{\sigma_{kb}^{**}-}^* - X_{\sigma_{kb}^{**}-}^*, \mathbf{n} \left(X_{\sigma_{kb}^{**}-}^* \right) \right\rangle \right|}{\left| Y_{\sigma_{kb}^{**}-}^* - X_{\sigma_{kb}^{**}-}^* \right|} \leq c_2 |Y_{\sigma_{(k-1)b}^{**}}^* - X_{\sigma_{(k-1)b}^{**}}^*|^{\beta_1} \right\}, \\ &\begin{cases} Y_{\sigma_{kb}^{**}}^* = Y_{\sigma_{kb}^{**}-}^* \\ Y_{\sigma_{kb}^{**}}^* = X_{\sigma_{kb}^{**}-}^* \\ + \pi_{X_{\sigma_{kb}^{**}-}^*} (Y_{\sigma_{kb}^{**}-}^* - X_{\sigma_{kb}^{**}-}^*) \frac{\left| X_{\sigma_{kb}^{**}-}^* - Y_{\sigma_{kb}^{**}-}^* \right| \vee \left| X_{\sigma_{(k-1)b}^{**}-}^* - Y_{\sigma_{(k-1)b}^{**}-}^* \right|}{\left| \pi_{X_{\sigma_{kb}^{**}-}^*} (Y_{\sigma_{kb}^{**}-}^* - X_{\sigma_{kb}^{**}-}^*) \right|} \end{cases} \begin{array}{l} \text{on } A_k \cap F_k, \\ \text{otherwise.} \end{array}\end{aligned} \quad (3.73)$$

Let $\{(X_t^*, Y_t^*), t \in [\sigma_{kb}^{**}, \sigma_{(k+1)b}^{**})\}$ be the solution to (1.1)-(1.2) with the initial conditions given by $X_{\sigma_{kb}^{**}}^* = X_{\sigma_{kb}^{**}-}^*$ and (3.73), and driven by Brownian motion $\{B_t, t \in [\sigma_{kb}^{**}, \sigma_{(k+1)b}^{**})\}$.

Whether $A_k \cap F_k$ holds or not, we have $|Y_{\sigma_{kb}^{**}}^* - X_{\sigma_{kb}^{**}}^*| \geq |Y_{\sigma_{kb}^{**}-}^* - X_{\sigma_{kb}^{**}-}^*|$ and

$$\frac{\left| \left\langle Y_{\sigma_{kb}^{**}}^* - X_{\sigma_{kb}^{**}}^*, \mathbf{n} \left(X_{\sigma_{kb}^{**}}^* \right) \right\rangle \right|}{\left| Y_{\sigma_{kb}^{**}}^* - X_{\sigma_{kb}^{**}}^* \right|} \leq c_2 |Y_{\sigma_{(k-1)b}^{**}}^* - X_{\sigma_{(k-1)b}^{**}}^*|^{\beta_1}. \quad (3.74)$$

If $\sigma_{kb}^{**} = \inf \left\{ t \geq \sigma_{(k-1)b}^{**} : L_t^{Y^*} - L_{\sigma_{(k-1)b}^{**}}^{Y^*} \geq b \right\}$ then we exchange the roles of X and Y in the above definitions.

In view of (3.74), we can apply Lemma 3.6 to the process $\{(X_t^*, Y_t^*), t \in [\sigma_{kb}^{**}, \sigma_{(k+1)b}^{**})\}$ to conclude that there exists an event $F_{k+1} \in \sigma((X_t^*, Y_t^*), t \in [\sigma_{kb}^{**}, \sigma_{(k+1)b}^{**}))$ such that, a.s. on the event $\{R_{\sigma_{kb}^{**}}^* \leq \varepsilon_1\}$,

$$\begin{aligned}\mathbb{P}\left(F_{k+1}^c \mid X_{\sigma_{kb}^{**}}^*, Y_{\sigma_{kb}^{**}}^*\right) &\leq (R_{\sigma_{kb}^{**}}^*)^{\beta_3}, \\ \mathbb{E}\left((V_{k+1}^* - V_k^*) \mathbf{1}_{F_{k+1}} \mid X_{\sigma_{kb}^{**}}^*, Y_{\sigma_{kb}^{**}}^*\right) &\geq c_3.\end{aligned} \quad (3.75)$$

Definition (3.73) implies that on F_{k+1}^c , we have $V_{k+1}^* \geq V_k^*$, so a.s. on the event $\{R_{\sigma_{kb}^{**}}^* \leq \varepsilon_1\}$,

$$\begin{aligned}\mathbb{E}\left(V_{k+1}^* - V_k^* \mid X_{\sigma_{kb}^{**}}^*, Y_{\sigma_{kb}^{**}}^*\right) &= \mathbb{E}\left((V_{k+1}^* - V_k^*) \mathbf{1}_{F_{k+1}} \mid X_{\sigma_{kb}^{**}}^*, Y_{\sigma_{kb}^{**}}^*\right) + \mathbb{E}\left((V_{k+1}^* - V_k^*) \mathbf{1}_{F_{k+1}^c} \mid X_{\sigma_{kb}^{**}}^*, Y_{\sigma_{kb}^{**}}^*\right) \\ &\geq c_3 + 0 = c_3.\end{aligned} \quad (3.76)$$

Let $K_1 = \inf\{k \geq 0 : \sup_{t \in [\sigma_{kb}^{**}, \sigma_{(k-1)b}^{**}]} R_t^* \geq \varepsilon_1\}$ and $\tilde{V}_k = V_{k \wedge K_1}^*$. It follows from (3.72) and (3.76) that $\{\tilde{V}_k, k \geq 0\}$ is a submartingale. Thus, \tilde{V}_k cannot converge to $-\infty$ with positive probability.

For a fixed j , we will estimate the number of k such that $\tilde{V}_k \in [j, j+1]$.

Let $c_4 = \sup_{x, y \in \overline{D}} \log |x - y|$ and note that $c_4 < \infty$. We will argue that for any $c_5 < c_4$, one can choose $\varepsilon_1 > 0$ so small that if $|x - y| \leq \varepsilon_1$ then $\mathbb{E}^{x, y} \tilde{V}_\infty \leq c_5$. Let $S = \inf\{t \geq 0 : R_t^* \geq \varepsilon_1\}$. By the proof of Lemma 3.4 in [3] (see the 2-dimensional version in Lemma 3.8 of [4]),

$$R_{\sigma_{K_1 b}^{**}}^* \leq \varepsilon_1 \exp(c_6((\sigma_{K_1 b}^{**} - S) + L_{\sigma_{K_1 b}^{**}}^Y - L_S^Y)) \leq \varepsilon_1 \exp(c_6(b + b)).$$

It follows that, for small ε_1 ,

$$\mathbb{E} \tilde{V}_\infty = \mathbb{E} \tilde{V}_{K_1} \leq \mathbb{E}(\log \varepsilon_1 + 2c_6 b) \leq c_5. \quad (3.77)$$

Consider any $c_5 \leq c_4$, assume that $\log \varepsilon_1 \leq c_5$ and fix an integer $j \leq c_5$. Let $U_1 = 0$ and

$$\begin{aligned} \hat{U}_k &= \inf\{n \geq U_k : \tilde{V}_n \notin [j-1, j+2]\}, \quad k \geq 1, \\ U_k &= \inf\{n \geq \hat{U}_k : \tilde{V}_n \in [j, j+1]\}, \quad k \geq 2, \\ K_2^j &= \sup\{k : U_k < \infty\}, \end{aligned}$$

with the convention that $\inf \emptyset = \infty$. The random variable K_2^j is bounded above by the sum of the number of upcrossings of the interval $[j-1, j]$ and the number of downcrossings of the interval $[j+1, j+2]$ by the process \tilde{V}_k . By the upcrossings and downcrossings inequalities, in view of (3.77),

$$\mathbb{E} K_2^j \leq 2(\mathbb{E} \tilde{V}_\infty - j + 2) \leq 2(c_5 - j + 2). \quad (3.78)$$

Let k_0 be the smallest integer greater than $3/c_1$, where c_1 is the constant in the statement of Lemma 3.4. Suppose that $\tilde{V}_{U_k} \in [j, j+1]$ for some $k \geq 0$. Then, using Lemma 3.4 and the strong Markov property at the stopping times σ_{nb}^{**} , $n = U_k, U_k + 1, \dots$, we see that for some $p_2 > 0$ and $p_3 := p_2^{k_0+1}$,

$$\mathbb{P}(\tilde{V}_{n+1} - \tilde{V}_n \geq c_2, n = U_k, U_{k+1}, \dots, U_{k+k_0}) \geq p_2^{k_0+1} = p_3.$$

If the event in the last formula occurs then the process \tilde{V} will leave the interval $[j-1, j+2]$ in at most $k_0 + 1$ steps so $\hat{U}_k - U_k \leq k_0 + 1$ in this case. If the process \tilde{V} does not leave $[j-1, j+2]$ in $k_0 + 1$ steps then we apply the same argument again, this time using stopping times $U_{k+k_0+2}, \dots, U_{k+2k_0+2}$. By induction, $(\hat{U}_k - U_k)/(k_0 + 1)$ is majorized by a geometric random variable with mean $1/p_3$. Hence, $\mathbb{E}(\hat{U}_k - U_k) \leq (k_0 + 1)/p_3$. Let K_3^j be the number of k such that $\tilde{V}_k \in [j, j+1]$. We combine the last formula with (3.78) to see that

$$\mathbb{E} K_3^j \leq 2(c_5 - j + 2)(k_0 + 1)/p_3. \quad (3.79)$$

This, (3.69) and (3.75) yield,

$$\begin{aligned}
\mathbb{P}\left(\bigcup_{k \geq 1} F_k^c\right) &\leq \mathbb{E}\left(\sum_{k \geq 1} \mathbf{1}_{F_k^c}\right) = \mathbb{E}\mathbf{1}_{F_1^c} + \sum_{k \geq 2} \mathbb{E}\mathbf{1}_{F_k^c} \\
&\leq p_1 + \sum_{j \leq c_5} \mathbb{E}\left(\sum_{\tilde{V}_{k-1} \in [j, j+1]} \mathbb{E}(\mathbf{1}_{F_k^c} \mid \tilde{V}_{k-1} \in [j, j+1])\right) \\
&\leq p_1 + \sum_{j \leq c_5} \mathbb{E}\left(\sum_{\tilde{V}_{k-1} \in [j, j+1]} e^{j\beta_3}\right) \\
&\leq p_1 + \sum_{j \leq c_5} e^{j\beta_3} 2(c_5 - j + 2)(k_0 + 1)/p_3.
\end{aligned}$$

By (3.79) and Lemma 3.3, for some $\beta_4 > 0$,

$$\begin{aligned}
\mathbb{P}\left(\bigcup_{k \geq 1} A_k^c\right) &\leq \mathbb{E}\left(\sum_{k \geq 1} \mathbf{1}_{A_k^c}\right) = \sum_{k \geq 1} \mathbb{E}\mathbf{1}_{A_k^c} \\
&= \sum_{j \leq c_5} \mathbb{E}\left(\sum_{\tilde{V}_{k-1} \in [j, j+1]} \mathbb{E}(\mathbf{1}_{A_k^c} \mid \tilde{V}_{k-1} \in [j, j+1])\right) \\
&\leq \sum_{j \leq c_5} \mathbb{E}\left(\sum_{\tilde{V}_{k-1} \in [j, j+1]} e^{j\beta_4}\right) \\
&\leq \sum_{j \leq c_5} e^{j\beta_4} 2(c_5 - j + 2)(k_0 + 1)/p_3.
\end{aligned}$$

We combine the last two estimates to obtain

$$\mathbb{P}\left(\bigcup_{k \geq 1} A_k^c \cup F_k^c\right) \leq \sum_{j \leq c_5} e^{j\beta_2} 2(c_5 - j + 2)(k_0 + 1)/p_3 + p_1 + \sum_{j \leq c_5} e^{j\beta_3} 2(c_5 - j + 2)(k_0 + 1)/p_3. \quad (3.80)$$

Consider an arbitrarily small $p_4 > 0$. The probability p_1 in (3.69) may be chosen to be smaller than $p_4/2$. We can make the two sums in (3.80) smaller than $p_4/2$ if we take $c_5 > -\infty$ sufficiently small. This we can do, as discussed earlier in the proof, by letting $\varepsilon_1 > 0$ be sufficiently small. Hence, if $\varepsilon_1 > 0$ is sufficiently small then

$$\mathbb{P}\left(\bigcup_{k \geq 1} A_k^c \cup F_k^c\right) \leq p_4. \quad (3.81)$$

Let

$$T_a^R = \inf\{t \geq 0 : R_t = a\}, \quad T_{-\infty}^R = \lim_{a \rightarrow -\infty} T_a^R.$$

Recall that \tilde{V}_k does not converge to $-\infty$ at a finite or infinite time, a.s. If all events $A_k \cap F_k$, $k \geq 1$, hold then $X_t^* = X_t$ and $Y_t^* = Y_t$ for all $t \geq 0$. This and (3.81) imply that for any $p_4 > 0$ there exists $\varepsilon_1 > 0$ such that for any $x, y \in \overline{D}$, $x \neq y$, we have $\mathbb{P}^{x,y}(T_{\varepsilon_1}^R < T_{-\infty}^R) \geq 1 - p_4$.

The process R_t takes values in the extended real line $[-\infty, \infty)$. If we endow this set with the natural topology then R_t is continuous for all $t \geq 0$, a.s., because processes X_t and Y_t are continuous.

Suppose that for some $x \neq y$, $p_5 := \mathbb{P}^{x,y}(T_{-\infty}^R < \infty) > 0$. We will show that this assumption leads to a contradiction. For $j \geq 1$, let $S_j = \inf\{t \geq 0 : R_t \leq 2^{-j}\}$ and

$$G_j = \left\{ \inf\{t \geq S_j : R_t = \varepsilon_1\} < \inf\{t \geq S_j : R_{t-} = -\infty\} \right\}.$$

If $T_{-\infty}^R < \infty$ then $S_j < \infty$ for all j . It follows from the strong Markov property applied at S_j that $\mathbb{P}^{x,y}(\{S_j < \infty\} \cap G_j) \geq p_5(1 - p_4)$. Since $\{S_{j+1} < \infty\} \cap G_{j+1} \subset \{S_j < \infty\} \cap G_j$, we have $\mathbb{P}^{x,y}\left(\bigcap_{j \geq 1}(\{S_j < \infty\} \cap G_j)\right) \geq p_5(1 - p_4) > 0$. If the event $\bigcap_{j \geq 1}(\{S_j < \infty\} \cap G_j)$ holds then R has a discontinuity at $T_{-\infty}^R$. Since R is continuous a.s., we have a contradiction which proves that for any $x \neq y$, $\mathbb{P}^{x,y}(T_{-\infty}^R < \infty) = 0$.

Now suppose that $p_6 := P(\lim_{t \rightarrow \infty} R_t = -\infty) > 0$. If $\lim_{t \rightarrow \infty} R_t = -\infty$ then $S_j < \infty$ for all j . We can argue as above to show that

$$\mathbb{P}^{x,y}\left(\left\{\lim_{t \rightarrow \infty} R_t = -\infty\right\} \cap \bigcap_{j \geq 1}(\{S_j < \infty\} \cap G_j)\right) \geq p_6(1 - p_4) > 0.$$

If the event $\bigcap_{j \geq 1}(\{S_j < \infty\} \cap G_j)$ holds then $\limsup_{t \rightarrow \infty} R_t > 0$. We have a contradiction which proves that for any $x \neq y$, $\mathbb{P}^{x,y}(\lim_{t \rightarrow \infty} R_t = -\infty) = 0$. \square

4. THE SIGN OF THE LYAPUNOV EXPONENT

This section is devoted to the calculation of the “Lyapunov exponent” for the exterior of a three-dimensional ball. In our model, the Lyapunov exponent is represented by $1 + \lambda_\rho$ where λ_ρ is defined in Theorem 4.1 (ii). This is a three-dimensional analogue of an exponent defined in [4] for two-dimensional domains. The sign of this exponent—positive for the domain D —has the fundamental importance for this article.

Recall that H^x is the excursion law for X in D . For an excursion e and non-zero vector $\mathbf{v} \in \mathbb{R}^3$, we let $f_{\mathbf{v}}(e) = \log |\pi_{e(\zeta-)}(\mathbf{v})| - \log |\mathbf{v}|$. Note that $f_{\mathbf{v}}(e) \leq 0$. Let $D_2 = \mathbb{R}^3 \setminus \overline{\mathcal{B}(0, 1)}$ and let (\hat{L}_t, \hat{H}^x) be the exit system for reflected Brownian motion \hat{X} in D_2 .

Theorem 4.1. (i) For every $x \in \partial D_2$ and $\mathbf{v} \in \mathcal{T}_x \partial D_2$,

$$\hat{H}^x(f_{\mathbf{v}}) = \sqrt{2} - 1 - \log(1 + \sqrt{2}).$$

(ii) Let $\lambda_\rho = H^x(f_{\mathbf{v}}(e))$. We have uniformly in $x \in \partial D$ and $\mathbf{v} \in \mathcal{T}_x \partial D$,

$$\lim_{\rho \rightarrow \infty} \lambda_\rho = \lim_{\rho \rightarrow \infty} H^x(f_{\mathbf{v}}(e)) = \sqrt{2} + \log 2 - 2 - \log(1 + \sqrt{2}) \approx -0.774013.$$

Proof. (i) Let $\tau_A^X = \inf\{t \geq 0 : X_t \notin A\}$. Recall that $P_{D_2}^x$ denotes the distribution of Brownian motion starting from x and killed at the time $\tau_{D_2}^X$. Let μ_r denote the uniform probability distribution on the sphere $\mathcal{B}(0, r)$; we will abbreviate $\mu_1 = \mu$. An explicit formula for the harmonic measure in D_2 is given in [15, Thm. 3.1, p. 102]. That formula implies that

$$P_{D_2}^x(X(\tau_{D_2}^X -) \in dy) = a(x)|x - y|^{-3}\mu(dy),$$

for $x \in D_2$ and $y \in \partial\mathcal{B}(0, 1)$, where $a(x)$ is such that for $x, y \in \partial\mathcal{B}(0, 1)$,

$$\lim_{\delta \downarrow 0} \frac{P_{D_2}^{x+\delta\mathbf{n}(x)}(X(\tau_{D_2}^X -) \in dy)}{2\delta|x + \delta\mathbf{n}(x) - y|^{-3}\mu(dy)} = 1.$$

We use (2.2) to see that for $x, y \in \mathcal{B}(0, 1)$,

$$\hat{H}_{D_2}^x(e_{\zeta-} \in dy) = 2|x - y|^{-3}\mu(dy). \quad (4.1)$$

Note that by symmetry, $\hat{H}^x(f_{\mathbf{v}})$ does not depend on $x \in \partial D_2$ and $\mathbf{v} \in \mathcal{T}_x \partial D_2$, so fix some x and \mathbf{v} . We will express $\mu(dy)$ and $f_{\mathbf{v}}$ using spherical coordinates. Let α denote the angle between radii of $\mathcal{B}(0, 1)$ going to $x, y \in \mathcal{B}(0, 1)$. Let M_1 be the plane that contains \mathbf{v} and 0, and let M_2 be the plane that contains 0, x and y . Let β be the angle between M_1 and M_2 .

The uniform probability measure on the sphere $\partial\mathcal{B}(0, 1)$ can be represented as

$$\mu(dy) = (2\pi)^{-1}d\beta(1/2)\sin\alpha d\alpha. \quad (4.2)$$

We have $|x - y| = 2\sin(\alpha/2)$ so (4.1)-(4.2) yield

$$\begin{aligned} \hat{H}_{D_2}^x(e_{\zeta-} \in dy) &= 2(2\sin(\alpha/2))^{-3}(2\pi)^{-1}d\beta(1/2)\sin\alpha d\alpha \\ &= \frac{1}{16\pi}\sin\alpha\sin(\alpha/2)^{-3}d\alpha d\beta. \end{aligned} \quad (4.3)$$

It is elementary (although somewhat tedious) to check that

$$\frac{|\pi_y(\mathbf{v})|}{|\mathbf{v}|} = (\cos^2\beta + \sin^2\beta\cos^2\alpha)^{1/2}.$$

If $e_{\zeta-} = y$ then

$$f_{\mathbf{v}}(e_t) = \log|\pi_{e_t(\zeta-)}(\mathbf{v})| - \log|\mathbf{v}| = \log(\cos^2\beta + \sin^2\beta\cos^2\alpha)^{1/2}. \quad (4.4)$$

We combine this formula with (4.3) to see that

$$\hat{H}_{D_2}^x(f_{\mathbf{v}}(e_t)) = \int_0^{2\pi} \int_0^\pi \frac{1}{16\pi} \sin\alpha\sin(\alpha/2)^{-3} \log(\cos^2\beta + \sin^2\beta\cos^2\alpha)^{1/2} d\alpha d\beta.$$

Part (i) of the theorem follows from this formula and Lemma 4.2 below.

(ii) First, we will show that the harmonic measure in a spherical shell has a density very close to a constant, under some assumptions. Let $S(r, R) = \mathcal{B}(0, 1) \setminus \overline{\mathcal{B}(0, r)}$ denote the

spherical shell with center 0, inner radius r and outer radius R . Let $h(r, R; x, y)$ be the density of harmonic measure in $S(r, R)$; more precisely, let

$$h(r, R; x, y) = \frac{P_{S(r, R)}^x(X_{\tau_{S(r, R)}^x} \in dy)}{\mu_r(dy)},$$

for $x \in S(r, R)$ and $y \in \partial\mathcal{B}(0, r)$. For fixed r, R and y , the function $x \rightarrow h(r, R; x, y)$ is harmonic in $S(r, R)$. By the Harnack principle, there exists $c_1 > 0$ such that for any positive harmonic function f in $\mathcal{B}(0, 1)$, we have $c_1 < f(v)/f(z) < 1/c_1$ for all $v, z \in \mathcal{B}(0, 1/2)$. By scaling, for any $r > 0$ and for any positive harmonic function f in $\mathcal{B}(0, r)$, we have $c_1 < f(v)/f(z) < 1/c_1$ for all $v, z \in \mathcal{B}(0, r/2)$. We can find a finite number N such that there exist $x_k \in \partial\mathcal{B}(0, 2r)$, $k = 1, \dots, N$, such that $\partial\mathcal{B}(0, 2r) \subset \bigcup_{1 \leq k \leq N} \mathcal{B}(x_k, r/2)$. Then the standard chaining argument shows that for $R \geq 3r$ and every positive harmonic function f in $S(r, R)$, we have $c_1^N < f(v)/f(z) < 1/c_1^N$ for all $v, z \in \mathcal{B}(0, 2r)$. Let $c_2 = c_1^N$. Consider a large integer m . As a particular case of the last formula, we obtain that

$$c_2 < h(2^k, 2^m; x, y)/h(2^k, 2^m; v, y) < 1/c_2, \quad (4.5)$$

for $0 \leq k \leq m-2$, $y \in \partial\mathcal{B}(0, 1)$ and $x, v \in \partial\mathcal{B}(0, 2^{k+1})$. By the strong Markov property for Brownian motion applied at the hitting time of $\partial\mathcal{B}(0, 2^{k+1})$,

$$h(2^k, 2^m; x, y) = \int_{\partial\mathcal{B}(0, 2^{k+1})} h(2^k, 2^m; v, y) h(2^{k+1}, 2^m; x, v) \mu_{2^{k+1}}(dv),$$

for $0 \leq k \leq m-3$, $y \in \partial\mathcal{B}(0, 1)$ and $x \in \partial\mathcal{B}(0, 2^{k+2})$. This, (4.5) and Lemma 6.1 of [6] imply, using the same argument as at the end of the proof of Theorem 6.1 in [6], that for any $c_3 < 1$ arbitrarily close to 1 there exists m_0 such that for $m \geq m_0$,

$$c_3 < h(1, 2^m; x, y)/h(1, 2^m; v, y) < 1/c_3,$$

for $y \in \partial\mathcal{B}(0, 1)$ and $x, v \in \partial\mathcal{B}(0, 2^{m-1})$. By applying a rotation, we obtain the following variant of the above result. For any $c_3 < 1$ arbitrarily close to 1 there exists m_0 such that for $m \geq m_0$,

$$c_3 < h(1, 2^m; x, y)/h(1, 2^m; x, z) < 1/c_3, \quad (4.6)$$

for $y, z \in \partial\mathcal{B}(0, 1)$ and $x \in \partial\mathcal{B}(0, 2^{m-1})$.

Suppose that ρ used in the definition of D satisfies $2^{m+1} \leq \rho \leq 2^{m+2}$ for some $m \geq m_0$. Let

$$T_1 = 0,$$

$$U_k = \inf\{t > T_k : X_{t-} \in \partial S(1, 2^m)\}, \quad k \geq 1,$$

$$T_k = \inf\{t > U_{k-1} : X_t \in \partial\mathcal{B}(0, 2^{m-1})\}, \quad k \geq 2.$$

Then for $x \in \partial\mathcal{B}(0, 2^{m-1})$ and $y \in \partial D = \partial\mathcal{B}(0, 1)$,

$$\begin{aligned} P_D^x(X_{\tau_D^x-} \in dy) &= \sum_{k=1}^{\infty} P_D^x(X_{U_k} \in dy; X_{U_j} \in \partial\mathcal{B}(0, 2^m), j < k) \\ &= \sum_{k=1}^{\infty} E_D^x \left(P_D^{X_{T_k}}(X_{U_k} \in dy) \mathbf{1}_{\{X_{U_j} \in \partial\mathcal{B}(0, 2^m), j < k\}} \right) \\ &= \sum_{k=1}^{\infty} E_D^x \left(h(1, 2^m; X_{T_k}, y) dy \mathbf{1}_{\{X_{U_j} \in \partial\mathcal{B}(0, 2^m), j < k\}} \right). \end{aligned}$$

This and (4.6) imply that

$$c_3 < P_D^x(X_{\tau_D^x-} \in dy) / P_D^x(X_{\tau_D^x-} \in dz) < 1/c_3,$$

for $y, z \in \partial\mathcal{B}(0, 1)$ and $x \in \partial\mathcal{B}(0, 2^{m-1})$. This and the strong Markov property of excursion laws applied at the hitting time $T_{\partial\mathcal{B}(0, 2^{m-1})}$ of $\partial\mathcal{B}(0, 2^{m-1})$ can be used to show that

$$c_3 < H^x(e_{\zeta-} \in dy; T_{\partial\mathcal{B}(0, 2^{m-1})} < \zeta) / H^x(e_{\zeta-} \in dz; T_{\partial\mathcal{B}(0, 2^{m-1})} < \zeta) < 1/c_3, \quad (4.7)$$

for $x, y, z \in \partial\mathcal{B}(0, 1)$. This implies that for sufficiently large ρ , the density of $H^x(e_{\zeta-} \in dy; T_{\partial\mathcal{B}(0, 2^{m-1})} < \zeta)$ is arbitrarily close to a constant on ∂D .

The probability that 3-dimensional Brownian motion starting from $x + \delta \mathbf{n}(x)$, $x \in \partial\mathcal{B}(0, 1)$, will never return to $\partial\mathcal{B}(0, 1)$ is equal to $1 - (1 + \delta)^{-1}$. This and (2.2) imply that for any $c_4 > 0$ there exists m_1 such that for $m \geq m_1$ and $x \in \partial D$,

$$1 - c_4 < H^x(T_{\partial\mathcal{B}(0, 2^{m-1})} < \zeta) < 1 + c_4.$$

It follows from this and (4.7) that for any $c_5 > 0$ and sufficiently large ρ , we have for $x, y \in \partial D$,

$$1 - c_5 < H^x(e_{\zeta-} \in dy; T_{\partial\mathcal{B}(0, 2^{m-1})} < \zeta) / \mu(dy) < 1 + c_5. \quad (4.8)$$

We have by continuity of probability that

$$\lim_{m \rightarrow \infty} H^x(e_{\zeta-} \in dy; T_{\partial\mathcal{B}(0, 2^{m-1})} > \zeta) = \hat{H}^x(e_{\zeta-} \in dy). \quad (4.9)$$

Note that the above limit is monotone.

We have

$$\begin{aligned} H^x(f_{\mathbf{v}}(e)) &= \int_{\partial D} (\log |\pi_y(\mathbf{v})| - \log |\mathbf{v}|) H^x(e_{\zeta-} \in dy) \\ &= \int_{\partial D} (\log |\pi_y(\mathbf{v})| - \log |\mathbf{v}|) H^x(e_{\zeta-} \in dy; T_{\partial\mathcal{B}(0, 2^{m-1})} > \zeta) \\ &\quad + \int_{\partial D} (\log |\pi_y(\mathbf{v})| - \log |\mathbf{v}|) H^x(e_{\zeta-} \in dy; T_{\partial\mathcal{B}(0, 2^{m-1})} < \zeta). \end{aligned} \quad (4.10)$$

It follows from (4.9), monotone convergence theorem and part (i) of this theorem that

$$\begin{aligned} & \lim_{m \rightarrow \infty} \int_{\partial D} (\log |\pi_y(\mathbf{v})| - \log |\mathbf{v}|) H^x(e_{\zeta-} \in dy; T_{\partial \mathcal{B}(0, 2^{m-1})} > \zeta) \\ &= \int_{\partial D} (\log |\pi_y(\mathbf{v})| - \log |\mathbf{v}|) \widehat{H}^x(e_{\zeta-} \in dy) = \widehat{H}^x(\widehat{f}_{\mathbf{v}}) = \sqrt{2} - 1 - \log(1 + \sqrt{2}). \end{aligned} \quad (4.11)$$

We combine (4.2), (4.4), (4.8) and Lemma 4.2 (ii) below to obtain

$$\begin{aligned} & \lim_{m \rightarrow \infty} \int_{\partial D} (\log |\pi_y(\mathbf{v})| - \log |\mathbf{v}|) H^x(e_{\zeta-} \in dy; T_{\partial \mathcal{B}(0, 2^{m-1})} > \zeta) \\ &= \int_{\partial D} (\log |\pi_y(\mathbf{v})| - \log |\mathbf{v}|) \mu(dy) \\ &= \int_0^{2\pi} \int_0^\pi \frac{1}{\pi} \sin \alpha \log(\cos^2 \beta + \sin^2 \beta \cos^2 \alpha)^{1/2} d\alpha d\beta = \log 2 - 1. \end{aligned}$$

Part (ii) of the theorem follows from this formula, (4.10) and (4.11). \square

Lemma 4.2. *We have*

$$\begin{aligned} & \int_0^{2\pi} \int_0^\pi \frac{1}{16\pi} \sin \alpha \sin(\alpha/2)^{-3} \log(\cos^2 \beta + \sin^2 \beta \cos^2 \alpha)^{1/2} d\alpha d\beta \\ &= \sqrt{2} - 1 - \log(1 + \sqrt{2}), \end{aligned} \quad (4.12)$$

and

$$\begin{aligned} & \int_0^{2\pi} \int_0^\pi \frac{1}{\pi} \sin \alpha \log(\cos^2 \beta + \sin^2 \beta \cos^2 \alpha)^{1/2} d\alpha d\beta \\ &= \log 2 - 1. \end{aligned} \quad (4.13)$$

Proof. The integral in (4.12) is equal to

$$\begin{aligned} & \int_0^{2\pi} \int_0^\pi \frac{1}{16\pi} \sin \alpha \sin(\alpha/2)^{-3} \log(\cos^2 \beta + \sin^2 \beta \cos^2 \alpha)^{1/2} d\alpha d\beta \\ &= (4/32\pi) \int_0^{\pi/2} d\beta \int_0^\pi \frac{\sin \alpha}{((1 - \cos \alpha)/2)^{3/2}} \log(\cos^2 \beta + \sin^2 \beta \cos^2 \alpha) d\alpha \\ &= (1/\sqrt{8}\pi) \int_0^{\pi/2} d\beta \int_{-1}^1 (1 - u)^{-3/2} \log(\cos^2 \beta + \sin^2 \beta u^2) du \quad (u = \cos \alpha) \\ &= (1/\sqrt{2}\pi) \int_0^{\pi/2} d\beta \int_{-1}^1 \log(\cos^2 \beta + \sin^2 \beta u^2) d(1 - u)^{-1/2} \end{aligned}$$

$$\begin{aligned}
&= (1/\sqrt{2}\pi) \int_0^{\pi/2} d\beta \left((1-u)^{-1/2} \log(\cos^2 \beta + \sin^2 \beta u^2) \Big|_{u=-1}^{u=1} \right. \\
&\quad \left. - \int_{-1}^1 \frac{1}{\sqrt{1-u}} \frac{2u \sin^2 \beta}{u^2 \sin^2 \beta + \cos^2 \beta} du \right) \\
&= -(\sqrt{2}/\pi) \int_0^{\pi/2} d\beta \int_{-1}^1 \frac{1}{\sqrt{1-u}} \frac{u \sin^2 \beta}{u^2 \sin^2 \beta + \cos^2 \beta} du \\
&= -(\sqrt{2}/\pi) \int_0^{\pi/2} d\beta \int_0^{\sqrt{2}} \frac{(1-v^2) \sin^2 \beta}{v((v^2-1)^2 \sin^2 \beta + \cos^2 \beta)} 2v dv \quad (v = \sqrt{1-u}) \\
&= -\frac{2\sqrt{2}}{\pi} \int_0^{\pi/2} d\beta \int_0^{\sqrt{2}} \frac{(1-v^2) \sin^2 \beta}{v^4 \sin^2 \beta - 2v^2 \sin^2 \beta + 1} dv \\
&= -\frac{2\sqrt{2}}{\pi} \int_0^{\sqrt{2}} (1-v^2) dv \int_0^{\sqrt{2}} \frac{1}{v^4 - 2v^2 + \frac{1}{\sin^2 \beta}} d\beta \\
&= -\frac{2\sqrt{2}}{\pi} \int_0^{\sqrt{2}} (1-v^2) dv \int_{+\infty}^0 \frac{1}{v^4 - 2v^2 + 1 + y^2} \cdot \frac{-1}{1+y^2} dy \quad (y = \cot \beta) \\
&= -\frac{2\sqrt{2}}{\pi} \int_0^{\sqrt{2}} (1-v^2) dv \int_0^{\infty} \frac{1}{(1-v^2)^2 + y^2} \cdot \frac{1}{1+y^2} dy \\
&= -\frac{2\sqrt{2}}{\pi} \int_0^{\sqrt{2}} (1-v^2) dv \int_0^{\infty} \frac{1}{(1-v^2)^2 - 1} \left(\frac{1}{1+y^2} - \frac{1}{(1-v^2)^2 + y^2} \right) dy \\
&= -\frac{2\sqrt{2}}{\pi} \int_0^{\sqrt{2}} \frac{1-v^2}{v^4 - 2v^2} dv \left(\arctan y - \frac{1}{|1-v^2|} \arctan \left(\frac{y}{|1-v^2|} \right) \right) \Big|_{y=0}^{y=\infty} \\
&= -\frac{2\sqrt{2}}{\pi} \int_0^{\sqrt{2}} \frac{1-v^2}{v^2(v^2-2)} \frac{\pi}{2} \left(1 - \frac{1}{|1-v^2|} \right) dv \\
&= -\sqrt{2} \int_0^1 \frac{1-v^2}{v^2(v^2-2)} \frac{(1-v^2)-1}{1-v^2} dv - \sqrt{2} \int_1^{\sqrt{2}} \frac{1-v^2}{v^2(v^2-2)} \frac{(v^2-1)-1}{1-v^2} dv \\
&= -\sqrt{2} \int_0^1 \frac{-1}{v^2-2} dv - \sqrt{2} \int_1^{\sqrt{2}} \frac{-1}{v^2} dv \\
&= -\sqrt{2} \int_0^1 \frac{1}{2\sqrt{2}} \left(\frac{1}{v+\sqrt{2}} - \frac{1}{v-\sqrt{2}} \right) dv - \sqrt{2} \frac{1}{v} \Big|_{v=1}^{v=\sqrt{2}} \\
&= -\frac{1}{2} \log \left| \frac{v+\sqrt{2}}{v-\sqrt{2}} \right| \Big|_{v=0}^{v=1} - 1 + \sqrt{2} \\
&= -\frac{1}{2} \log \left| \frac{1+\sqrt{2}}{1-\sqrt{2}} \right| - 1 + \sqrt{2} \\
&= -\frac{1}{2} \log \left((1+\sqrt{2})^2 \right) - 1 + \sqrt{2} \\
&= \sqrt{2} - 1 - \log(1+\sqrt{2}).
\end{aligned}$$

The integral in (4.13) is equal to

$$\begin{aligned}
& \int_0^{2\pi} \frac{d\beta}{2\pi} \int_0^\pi (1/2) \sin \alpha \log((\cos^2 \beta + \sin^2 \beta \cos^2 \alpha)^{1/2}) d\alpha \\
&= 4 \int_0^{\pi/2} \frac{d\beta}{2\pi} \int_0^\pi (1/4) \sin \alpha \log(\cos^2 \beta + \sin^2 \beta \cos^2 \alpha) d\alpha \\
&= - \int_0^{\pi/2} \frac{d\beta}{2\pi} \int_1^{-1} \log(\cos^2 \beta + u^2 \sin^2 \beta) du \quad (u = \cos \alpha) \\
&= 2 \int_0^{\pi/2} \frac{d\beta}{2\pi} \int_0^1 \log(\cos^2 \beta + u^2 \sin^2 \beta) du \\
&= \frac{1}{\pi} \int_0^{\pi/2} d\beta \left(u \log(\cos^2 \beta + u^2 \sin^2 \beta) \Big|_{u=0}^{u=1} - \int_0^1 \frac{2u^2 \sin^2 \beta}{\cos^2 \beta + u^2 \sin^2 \beta} du \right) \\
&= -\frac{2}{\pi} \int_0^1 u^2 du \int_0^{\pi/2} \frac{\sin^2 \beta}{\cos^2 \beta + u^2 \sin^2 \beta} d\beta \\
&= -\frac{2}{\pi} \int_0^1 u^2 du \int_0^{\pi/2} \frac{1}{\cot^2 \beta + u^2} d\beta \\
&= -\frac{2}{\pi} \int_0^1 u^2 du \int_0^{\pi/2} \frac{1}{\cot^2 \beta + u^2} \cdot \frac{-1}{\csc^2 \beta} (-\csc^2 \beta) d\beta \\
&= -\frac{2}{\pi} \int_0^1 u^2 du \int_{+\infty}^0 \frac{1}{y^2 + u^2} \cdot \frac{-1}{y^2 + 1} dy \quad (y = \cot \beta) \\
&= -\frac{2}{\pi} \int_0^1 u^2 du \int_0^\infty \left(\frac{1}{y^2 + 1} - \frac{1}{y^2 + u^2} \right) \frac{1}{u^2 - 1} dy \\
&= -\frac{2}{\pi} \int_0^1 \frac{u^2}{u^2 - 1} du \left(\arctan y - \frac{1}{u} \arctan(y/u) \right) \Big|_{y=0}^{y=\infty} \\
&= -\frac{2}{\pi} \int_0^1 \frac{u^2}{u^2 - 1} \cdot \frac{\pi}{2} \left(1 - \frac{1}{u} \right) du \\
&= - \int_0^1 \frac{u^2}{u^2 - 1} \cdot \frac{u - 1}{u} du \\
&= - \int_0^1 \left(1 - \frac{1}{u + 1} \right) du \\
&= - \left(1 - \log(u + 1) \right) \Big|_{u=0}^{u=1} \\
&= \log 2 - 1.
\end{aligned}$$

□

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